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Introduction

Linear programming is concerned with problems in which a linear objective function in terms of decision variables is to be optimized (i.e., either minimized or maximized) while a set of linear equations, inequalities, and sign restrictions are imposed on the decision variables as requirements. Linear programming is a quite young and yet very active branch of applied mathematics. The wide applicability of linear programming models and the evolving mathematical theory and computational methodology under these models have attracted an immense amount of interest from both practitioners and academicians in the past five decades. In a recent survey of Fortune 500 companies, 85% of those responding said that they had used linear programming.

In this chapter, we briefly review the history of linear programming in Section 1, introduce linear programming problems in Section 2, and give linear programming examples in Section 3. The layout of the book is discussed in the final section.

1.1 HISTORY OF LINEAR PROGRAMMING

The linear programming problem was first conceived by G. B. Dantzig around 1947 while he was working as a Mathematical Advisor to the United States Air Force Comptroller on developing a mechanized planning tool for a deployment, training, and logistical supply *program*. The work led to his 1948 publication, "Programming in a Linear Structure." The name "linear programming" was coined by T. C. Koopmans and Dantzig in the summer of 1948, and an effective "simplex method" for solving linear programming problems was proposed by Dantzig in 1949. In the short period between 1947 and 1949, a major part of the foundation of linear programming was laid. As early as 1947, Koopmans began pointing out that linear programming provided an excellent framework for the analysis of classic economic theories.

Linear programming was not, however, born overnight. Prior to 1947, mathematicians had studied systems of linear inequalities, the core of the mathematical theory of linear programming. The investigation of such systems can be traced to Fourier's work in 1826. Since then, quite a few mathematicians have considered related subjects. In particular, the optimality conditions for functions with inequality constraints in the finite-dimensional case appeared in W. Karush's master's thesis in 1939, and various special cases of the fundamental duality theorem of linear programming were proved by others. Also, as early as 1939, L. V. Kantorovich pointed out the practical significance of a restricted class of linear programming models for production planning and proposed a rudimentary algorithm for their solution. Unfortunately, Kantorovich's work remained neglected in the U.S.S.R. and unknown elsewhere until long after linear programming had been well established by G. B. Dantzig and others.

Linear programming kept evolving in the 1950s and 1960s. The theory has been enriched and successful applications have been reported. In 1975, the topic came to public attention when the Royal Sweden Academy of Sciences awarded the Nobel Prize in economic science to L. V. Kantorovich and T. C. Koopmans "for their contributions to the theory of optimum allocation of resources." Yet another dramatic development in linear programming came to public attention in 1979: L. G. Khachian proved that the so-called "ellipsoid method" of N. Z. Shor, D. B. Yudin, and A. S. Nemirovskii, which differs radically from the simplex method, could outperform the simplex method in theory. Unlike the simplex method, which might take an exponential number of iterations to reach an optimal solution, the ellipsoid method finds an optimal solution of a linear programming problem in a *polynomial-time* bound. Newspapers around the world published reports of this result as if the new algorithm could solve the most complicated and large-scale resource allocation problems in no time. Unfortunately, the theoretic superiority of the ellipsoid method could not be realized in practical applications.

In 1984, a real breakthrough came from N. Karmarkar's "projective scaling algorithm" for linear programming. The new algorithm not only outperforms the simplex method in theory but also shows its enormous potential for solving very large scale practical problems. Karmarkar's algorithm is again radically different from the simplex method—it approaches an optimal solution from the interior of the feasible domain. This interior-point approach has become the focal point of research interests in recent years. Various theoretic developments and real implementations have been reported, and further results are expected.

1.2 THE LINEAR PROGRAMMING PROBLEM

In this section, we first introduce a linear programming problem in its *standard form*, then discuss the embedded assumptions of linear programming, and finally show a mechanism to convert any general linear programming problem into the standard form.

1.2.1 Standard-Form Linear Program

A *standard-form* linear programming problem can be described as follows:

$$\begin{aligned}
\text{Minimize}\quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to}\quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
& x_1, x_2, \ldots x_n \geq 0
\end{aligned} \tag{1.1}$$

In which, $x_1, x_2, \ldots, x_n$ are nonnegative *decision variables* to be determined and $c_1, c_2, \ldots, c_n$ are *cost coefficients* associated with the decision variables such that the *objective function* $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized. Moreover, $\sum_{j=1}^{n} a_{ij}x_j = b_i$ denotes the ith *technological constraint* for $i = 1, \ldots, m$, where a_{ij}, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, are the *technological coefficients* and b_i, for $i = 1, \ldots, m$, are the *right-hand-side coefficients.*

A linear programming problem (in standard form) is to find a specific nonnegative value for each decision variable such that the objective function achieves its minimum at this particular solution while all the technological constraints are satisfied.

If we denote $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mathbf{c} = (c_1, \ldots, c_n)^T$, $\mathbf{b} = (b_1, \ldots, b_m)^T$, and $\mathbf{A} =$ matrix of (a_{ij}), then the above linear programming problem can be written in matrix notation as follows:

$$\begin{aligned}
\text{Minimize}\quad & \mathbf{c}^T\mathbf{x} \\
\text{subject to}\quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned} \tag{1.2}$$

1.2.2 Embedded Assumptions

In order to represent an optimization problem as a linear programming problem, implicitly we make the following assumptions:

1. *Proportionality assumption:* For each decision variable x_j, for $j = 1, \ldots, n$, its contribution to the objective function z and to each constraint

   $$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad \text{for} \qquad i = 1, \ldots, m,$$

   is directly proportional to its value. There are no economies of returns to scale or discounts at all. To be more specific, one unit of variable x_j contributes c_j units in the objective function and a_{ij} units in the ith constraint, and two units of variable

x_j contribute exactly $2c_j$ units in the objective function and $2a_{ij}$ units in the ith constraint. No set-up cost for starting the activity is realized.

2. *Additivity assumption:* The contribution to the objective function or any technological constraint of any decision variable is independent of the values of other decision variables. There are no interaction or substitution effects among the decision variables. The total contribution is the sum of the individual contributions of each decision variable.
3. *Divisibility assumption:* Each decision variable is allowed to assume any fractional value. In other words, noninteger values for the decision variables are permitted.
4. *Certainty assumption:* Each parameter (the cost coefficient c_j, the technological coefficient a_{ij}, and the right-hand-side coefficient b_i) is known with certainty. No probabilistic or stochastic element is involved in a linear programming problem.

It is clearly seen that a nonlinear function could violate the proportionality assumption and additivity assumption, an integer requirement on the decision variables could ruin the divisibility assumption, and a probabilistic scenario could rule out the certainty assumption. Although the embedded assumptions seem to be very restrictive, linear programming models are nonetheless among the most widely used models today.

1.2.3 Converting to Standard Form

The standard form of linear program deals with a linear minimization problem with nonnegative decision variables and linear equality constraints. In general, a linear program is a problem of minimizing or maximizing a linear objective function with restricted or unrestricted decision variables in the presence of linear equality and/or inequality constraints. Here we introduce a mechanism to convert any general linear program into the standard form.

Linear Inequalities and Equations. A linear inequality can be easily converted into an equation. If the ith technological constraint has the form

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

we can add a nonnegative *slack variable* $s_i \geq 0$ to make a linear equation

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$$

Similarly, if the ith technological constraint has the form

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$$

we can subtract a nonnegative *surplus variable* $e_i \geq 0$ to make a linear equation

$$\sum_{j=1}^{n} a_{ij}x_j - e_i = b_i$$

On the other hand, a linear equation $\sum_{j=1}^{n} a_{ij}x_j = b_i$ can be converted into a pair of inequalities, namely,

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij}x_j \geq b_i$$

Restricted and Unrestricted Variables. The decision variables in a standard-form linear program are required to be nonnegative. If a variable is restricted to be $x_j \geq l_j$, we can replace x_j by $\bar{x}_j + l_j$ and require the new variable $\bar{x}_j \geq 0$. Similarly, if a variable is restricted to be $x_j \leq u_j$, we can replace x_j by $u_j - \bar{x}_j$ and require the new variable $\bar{x}_j \geq 0$.

As to an unrestricted variable $x_j \in R$, we can replace it by $\bar{x}_j - \hat{x}_j$ with two new variables $\bar{x}_j \geq 0$ and $\hat{x}_j \geq 0$. Also note that, if $x_1, \ldots, x_k$ are a group of unrestricted variables, we need to introduce only $k+1$ new variables, $\bar{x}_1, \ldots, \bar{x}_k$ and $\hat{x}$ such that x_j is replaced by $\bar{x}_j - \hat{x}$, for $j = 1, \ldots, k$ with $\bar{x}_j \geq 0$ and $\hat{x} \geq 0$.

Maximization and Minimization. In case our objective is to maximize a linear function, instead of minimizing, note that over any given region,

$$\text{maximum}\left(\sum_{j=1}^{n} c_jx_j\right) = -\text{minimum}\left(\sum_{j=1}^{n} -c_jx_j\right)$$

Therefore, we simply multiply the cost coefficients by -1 to convert a maximization problem into a minimization problem. But, once the minimum of the new problem is found, remember to multiply the minimum value by -1 for the original maximum.

Canonical-Form Linear Program. In addition to the standard form, linear programming problems are also commonly represented in the following *canonical form*:

$$\begin{aligned} &\text{Minimize} && \sum_{j=1}^{n} cjx_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij}x_j \geq b_i, && \text{for } i = 1, \ldots, m \\ & && x_j \geq 0, && \text{for } j = 1, \ldots, n \end{aligned}$$

1.3 EXAMPLES OF LINEAR PROGRAMMING PROBLEMS

Modeling a problem is always an art. Although linear programming has long proved its merit as an effective model of numerous applications, still there is no fixed rule of modeling. In this section we present some classic examples of situations that have natural formulations, from which we see that a general practice is to define decision variables first. Each decision variable is associated with a certain activity of interest, and the value of a decision variable may represent the level of the associated activity. Once the decision variables are defined, the objective function usually represents the gain or loss of taking these activities at different levels, and each technological constraint depicts certain interrelationships among those activities. However, many sophisticated applications go far beyond the general practice.

Example 1.1 *The diet problem*

Suppose n different food items are available at the market and the selling price for the jth food is c_j per unit. Moreover, there are m basic nutritional ingredients for the human body and a minimum of b_i units of the ith ingredient are required to achieve a balanced diet for good health. In addition, a study shows that each unit of the jth food contains a_{ij} units of the ith nutritional ingredient. A dietitian of a large group may face a problem of determining the most economical diet that satisfies the basic minimum nutritional requirements for good health.

Since the activity of interest here is to determine the quantity of each food in the diet, we define x_j to be the number of units of food j in the diet, for $j = 1, \ldots, n$. Then the problem is to determine x_j's which minimize the total cost

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the nutritional requirements

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{gathered}$$

and the nonnegativity constraints

$$x_1 \geq 0, \quad x_2 \geq 0, \ldots, \quad x_n \geq 0$$

By subtracting a nonnegative surplus variable for each constraint, we have a linear programming problem in its standard form:

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{j=1}^{n} c_jx_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - \bar{x}_i = b_i, & \text{for } i = 1, \ldots, m \\ & x_j \geq 0, \bar{x}_i \geq 0, & \text{for } j = 1, \ldots, n, \ i = 1, \ldots, m \end{array}$$

Example 1.2 *The transportation problem*

A moving company is contracted to ship certain product from m sources to n destinations. There are a_i units of product stored at the ith source, for $i = 1, \ldots, m$, and a minimum of b_j units of product are required to be received at the jth destination, for $j = 1, \ldots, n$. Suppose the customer is willing to pay a price of c_{ij} for moving one unit of product from source i to destination j and the moving company is interested in fulfilling the contract with a maximum earning.

Since the major activity of interest is to ship the product from a source to a destination, we define x_{ij} to be the number of units of product shipping from the ith source to the jth destination, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then the problem is to find x_{ij}'s which maximize the total earnings

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to the source constraints

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \qquad \text{for } i = 1, 2, \ldots, m$$

the destination constraints

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and the nonnegativity constraints

$$x_{ij} \geq 0, \qquad \text{for } i = 1, \ldots, m, \ j = 1, \ldots, n$$

By adding a nonnegative slack variable to each source constraint, subtracting a nonnegative surplus variable from each destination constraint, and multiplying the total earning by -1, we have a standard-form linear programming problem:

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} -c_{ij}x_{ij} & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} + \bar{x}_i = a_i, & \text{for } i = 1, 2, \ldots, m \\ & \displaystyle\sum_{i=1}^{m} x_{ij} - \hat{x}_j = b_j, & \text{for } j = 1, 2, \ldots, n \\ & x_{ij} \geq 0, \bar{x}_i \geq 0, \hat{x}_j \geq 0 & \text{for } i = 1, \ldots, m, \ j = 1, 2, \ldots, n \end{array}$$

To assure this problem has a feasible solution, the condition

$$\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$$

is, of course, assumed.

Example 1.3 *The warehousing problem*

A warehouse has a fixed capacity C. The manager of the warehouse buys and sells the stock of a certain commodity over a certain length of time to make profit. We break the time window into n periods (say one week per period) and assume that in the jth period the same unit price p_j holds for both purchase and sale. In addition, there is a unit cost r for holding stock for one period. The warehouse is empty at the beginning and is required to be empty at the end. How should the manager operate?

The major activities involve buying, selling, and holding the stock in each period. We define x_j to be the level of stock in the warehouse at the beginning of the jth period, y_j the amount bought during the period, and z_j the amount sold during the period. Then the manager tries to maximize his profit

$$\sum_{j=1}^{n}(p_j z_j - p_j y_j - r x_j)$$

subject to the inventory-balance constraints

$$x_{j+1} = x_j + y_j - z_j, \qquad \text{for } j = 1, \ldots, n-1$$

the warehouse-capacity constraints

$$x_j \leq C, \qquad \text{for } j = 1, \ldots, n$$

the boundary conditions

$$x_1 = 0$$

$$x_n + y_n - z_n = 0$$

and the nonnegativity constraints

$$x_j \geq 0, \quad y_j \geq 0, \quad z_j \geq 0, \qquad \text{for } j = 1, \ldots, n$$

After converting, we have a standard-form linear program:

$$\begin{aligned}
-\text{Minimize} \quad & \sum_{j=1}^{n}(-p_j z_j + p_j y_j + r x_j) \\
\text{subject to} \quad & x_j - x_{j+1} + y_j - z_j = 0, \qquad \text{for } j = 1, \ldots, n-1 \\
& x_j + \bar{x}_j = C, \qquad \text{for } j = 1, \ldots, n \\
& x_1 = 0 \\
& x_n + y_n - z_n = 0 \\
& x_j \geq 0, \quad \bar{x}_j \geq 0, \quad y_j \geq 0, \quad z_j \geq 0, \qquad \text{for } j = 1, \ldots, n
\end{aligned}$$

Example 1.4 *The cutting-stock problem*

A metal slitting company cuts master rolls of standard width w and length l into subrolls of smaller width but the same length l. Customers specify their orders in terms of the number of subrolls of different widths. The objective is to use a minimum number of master rolls to satisfy a set of customers' orders.

Suppose that there are m different widths specified by customers, say $w_1, w_2, \ldots, w_m$, and customers require b_i subrolls of width w_i, for $i = 1, \ldots, m$. For a master roll with width w (of course, $w_i \leq w$ for each i), there are many ways to cut it into subrolls. For example, subrolls of widths 3, 5, 7 are cut from a master roll of width 10. We can cut a master roll to produce three subrolls of width 3, zero subrolls of width 5, and zero subrolls of width 7; or cut to produce one subroll of width 3, zero subrolls of width 5, and one subroll of width 7; or cut to produce zero subrolls of width 3, two subrolls of width 5, and zero subrolls of width 7, …, etc. Each such way is called a feasible *cutting pattern*. Although the total number of all possible cutting patterns may become huge, the number of feasible cutting patterns is always finite, say n. If we let a_{ij} be the number of subrolls of width w_i obtained by cutting one master roll according to pattern j, then

$$\sum_{i=1}^{m} a_{ij} w_i \leq w$$

is required for the pattern to be feasible. Now define x_j to be the number of master rolls cut according to the jth feasible pattern, and the cutting-stock problem becomes an *integer linear programming problem:*

$$\begin{aligned}
\text{Minimize} \quad & \sum_{j=1}^{n} x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq bi \qquad \text{for } i = 1, \ldots, m \\
& x_j \geq 0 \qquad \text{for } j = 1, \ldots, n \\
& x_j\text{: integer} \qquad \text{for } j = 1, \ldots, n
\end{aligned}$$

If the integrality requirement on the x_j's is dropped, the problem becomes a linear programming problem.

1.4 MASTERING LINEAR PROGRAMMING

This is a book of linear programming and its extensions. The authors see three key elements in the mastering of linear programming, namely,

1. Intuitions generated by observing geometric interpretations.
2. Properties proven by manipulating algebraic expressions.
3. Algorithms validated by computer implementations.

The first step of learning is to "see" problems and have a feeling about them. In this way, we are led to understand the known properties and conjecture new ones. The second step is to translate geometric properties into algebraic expressions and to develop algebraic skills to manipulate them in proving new results. Once the problems are understood and basic results are obtained, the third step is to develop solution procedures.

Since the most important characteristic of a high-speed computer is its ability to perform repetitive operations very efficiently, linear programming algorithms are introduced in an *iterative scheme* and validated by computer implementations.

The basic philosophy of solving a linear programming problem via an iterative scheme is to start from a rough solution and successively improve the current solution until a set of desired optimality conditions are met. In this book, we treat the simplex method, the ellipsoid method, and Karmarkar's algorithm and its variants from this integrated iterative approach. The layout of the book is as follows. We provide simple geometry of linear programming in Chapter 2, introduce the classic simplex method in Chapter 3, and study the fascinating duality theory and sensitivity analysis in Chapter 4. From the complexity point of view, we further introduce Khachian's ellipsoid method in Chapter 5 and Karmarkar's algorithm in Chapter 6. The affine scaling algorithms, as variants of Karmarkar's algorithm, are the topics of Chapter 7. The insights of the interior-point methods are discussed in Chapter 8. Then we extend our horizon to touch on the convex quadratic programming in Chapter 9. Finally we wrap up the book by studying the computer implementation issues in Chapter 10.

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EXERCISES

1.1. Convert the following linear programming problems into standard form:

(a) Minimize $4x_1 + \sqrt{2}x_2 - 0.35x_3$

$$\begin{aligned} \text{subject to} \quad -0.001x_1 + 200x_2 &\geq 7\sqrt{261} \\ 7.07x_2 - 2.62x_3 &\leq -4 \\ x_1, x_3 &\geq 0 \end{aligned}$$

(b) Maximize $-3.1x_1 + 2\sqrt{2}x_2 - x_3$

$$\begin{aligned} \text{subject to} \quad 100x_1 - 20x_2 &= 7 \\ -11x_1 - 7\pi x_2 - 2x_3 &\leq 400 \\ x_1 \geq 20,\ x_2 \geq 0,\ x_3 &\geq -15 \end{aligned}$$

(c) Maximize $x_1 + 3x_2 - 2x_3$

$$\begin{aligned} \text{subject to} \quad -2 \leq 3x_1 - 5x_2 &\leq 15 \\ 11 \leq -5x_1 + 20x_2 &\leq 40 \\ x_2 \geq 0,\ x_3 &\leq 10 \end{aligned}$$

1.2. Consider a linear programming problem:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + 6x_2 + 8x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 = 5 \\ & 4x_1 + 6x_2 + 2x_3 = 12 \\ & x_2 \geq 0,\ x_3 \geq 0 \end{aligned}$$

(a) Convert this problem into its standard form.

(b) Can you find an equivalent linear programming problem with only two variables? [*Hint*: Eliminate x_1 from the constraints and replace it by $5-2x_2-x_3$ in the objective function.]

(c) Convert the equivalent linear program into standard form.

(d) Try to solve the problem.

1.3. Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & x_1^2 + x_2 + 4x_3 \\ \text{subject to} \quad & x_1^2 - x_2 = 0 \\ & 2x_2 + 4x_3 \geq 4 \\ & x_1 \geq 0, x_2 \geq 2, x_3 \geq 0 \end{aligned}$$

(a) Is this a linear programming problem?

(b) Can you solve this problem by finding an equivalent linear programming problem? [*Hint:* Use the first constraint.]

(c) Can you convert the equivalent linear programming problem into its standard form?

(d) Can you solve the linear program? the original problem?

1.4. Consider the following optimization problem:

$$\begin{aligned} \text{Minimize} \quad & |x_1| + 2|x_2| - |x_3| \\ \text{subject to} \quad & x_1 + x_2 - x_3 \leq 10 \\ & x_1 - 3x_2 + 2x_3 = 12 \end{aligned}$$

(a) Is this a linear programming problem?

(b) Can you convert it into a linear program in standard form? [*Hint*: For any real number x, we can find $u, v \geq 0$ such that $|x| = u + v$ and $x = u - v$.]

(c) Convert the following problem into a linear program in standard form:

$$\begin{aligned} \text{Minimize} \quad & |x_1 - 5| + |x_2 + 4| \\ \text{subject to} \quad & x_1 + x_2 \leq 10 \\ & x_1 - 3x_2 \geq 2 \end{aligned}$$

1.5. CHIPCO produces two kinds of memory chips (Chip-1 and Chip-2) for computer usage. The unit selling price is $15 for Chip-1 and $25 for Chip-2. To make one Chip-1, CHIPCO has to invest 3 hours of skilled labor, 2 hours of unskilled labor, and 1 unit of raw material. To make one Chip-2, it takes 4 hours of skilled labor, 3 hours of unskilled labor, and 2 units of raw material. The company has 100 hours of skilled labor, 70 hours of unskilled labor, and 30 units of raw material available. The sales contract signed by CHIPCO requires that at least 3 units of Chip-2 have to be produced and any fractional quantity is acceptable.

Can you formulate a linear program to help CHIPCO determine its optimal product mix?

1.6. *Assignment problem.* Five persons (A, B, C, D, E) are assigned to work on five different projects. The following table shows how long it takes for a specific person to finish a specific project:

	Project #				
	1	2	3	4	5
A	5	5	7	4	8
B	6	5	8	3	7
C	6	8	9	5	10
D	7	6	6	3	6
E	6	7	10	6	11

The standard wage is $60 per person per day. Suppose that one person is assigned to do one project and every project has to be covered by one person. Can you formulate this problem as an integer linear program?

1.7. INTER-TRADE company buys no-bland textile outlets from China, India, and the Philippines, ships to either Hong Kong or Taiwan for packaging and labeling, and then ships to the United States or France for sale. The transportation costs between sources and destinations can be read from the following table:

	China	India	Philippines	USA	France
Hong Kong	$50/ton	$90/ton	$70/ton	$150/ton	$180/ton
Taiwan	$60/ton	$95/ton	$50/ton	$130/ton	$200/ton

Suppose that INTER-TRADE purchased 60 tons of no-blands from China, 45 tons from India, and 30 tons from the Philippines. The U.S. market demands 80 tons of labeled products and the French market 55 tons. Assume that packaging and labeling do not change the weight of textile products.

(a) If both Hong Kong and Taiwan have unlimited packaging and labeling capacity, formulate a linear program to help INTER-TRADE minimize the shipping cost.

(b) If Hong Kong can process at most 60 tons of no-blands, what will be changed in your formulation?

(c) If Hong Kong can process at most 60 tons of no-blands and Taiwan can process at most 50 tons, what will happen to your formulation?

(d) Under condition (c), try to reduce the linear program to two independent transportation problems.