2

Geometry of Linear Programming

The intent of this chapter is to provide a geometric interpretation of linear programming. Once the underlying geometry is understood, we can follow intuitions to manipulate algebraic expressions in validating known results and developing new insights into linear programming. We shall stick to linear programs in standard form in this chapter. Some terminology and basic concepts will be defined before the fundamental theorem of linear programming is introduced. Motivations of the classic simplex method and the newly developed interior-point approach will then be discussed.

2.1 BASIC TERMINOLOGY OF LINEAR PROGRAMMING

Consider a linear programming problem in its standard form:

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{2.1}$$

where $\mathbf{c}$ and $\mathbf{x}$ are n-dimensional column vectors, $\mathbf{A}$ an $m \times n$ matrix, and $\mathbf{b}$ an m-dimensional column vector. Usually, $\mathbf{A}$ is called the *constraint matrix*, $\mathbf{b}$ the *right-hand-side vector,* and $\mathbf{c}$ the *cost vector.* Note that we can always assume that $\mathbf{b} \geq \mathbf{0}$, since for any component $b_i < 0$, multiplying a factor -1 on both sides of the ith constraint results in a new positive right-hand-side coefficient. Now we define $P = \{\mathbf{x} \in R^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ to be the *feasible domain* or *feasible region* of the linear program. When P is not void, the linear program is said to be *consistent.* For a consistent linear program with a

feasible solution $\mathbf{x}^* \in P$, if $\mathbf{c}^T\mathbf{x}^*$ attains the minimum value of the objective function $\mathbf{c}^T\mathbf{x}$ over the feasible domain P, then we say $\mathbf{x}^*$ is an *optimal solution* to the linear program. We also denote $P^* = \{\mathbf{x}^* \in P \mid \mathbf{x}^* \text{ is an optimal solution}\}$ as the *optimal solution set.*

Moreover, we say a linear program has a *bounded* feasible domain, if there exists a positive constant M such that for every feasible solution $\mathbf{x}$ in P, its Euclidean norm, $\|\mathbf{x}\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$, is less than or equal to M. On the other hand, if there exists a constant C such that $\mathbf{c}^T\mathbf{x} \geq C$ for each $\mathbf{x} \in P$, then we say the linear program is *bounded.* In this context, we know a linear program with bounded feasible domain must be bounded, but the converse statement needs not to be true.

Our immediate objective is to examine the geometry of the feasible domain P and the linear objective function $\mathbf{c}^T\mathbf{x}$ of a linear program.

2.2 HYPERPLANES, HALFSPACES, AND POLYHEDRAL SETS

A fundamental geometric entity occurring in linear optimization is the *hyperplane*

$$H = \{\mathbf{x} \in R^n \mid \mathbf{a}^T\mathbf{x} = \beta\} \tag{2.2}$$

whose description involves a nonzero n-dimensional column vector $\mathbf{a}$ and a scalar β. A hyperplane separates the whole space into two *closed halfspaces*

$$H_L = \{\mathbf{x} \in R^n \mid \mathbf{a}^T\mathbf{x} \leq \beta\} \tag{2.3}$$

and

$$H_U = \{\mathbf{x} \in R^n \mid \mathbf{a}^T\mathbf{x} \geq \beta\} \tag{2.4}$$

that intersect at the hyperplane H. Removing H results in two disjoint *open halfspaces*

$$H_L^i = \{\mathbf{x} \in R^n \mid \mathbf{a}^T\mathbf{x} < \beta\} \tag{2.5}$$

and

$$H_U^i = \{\mathbf{x} \in R^n \mid \mathbf{a}^T\mathbf{x} > \beta\} \tag{2.6}$$

We further define H to be the *bounding hyperplane* of H_L and H_L^i.

The defining vector $\mathbf{a}$ of hyperplane H is called the *normal* of H. Since, for any two vectors $\mathbf{y}$ and $\mathbf{z} \in H$,

$$\mathbf{a}^T(\mathbf{y} - \mathbf{z}) = \mathbf{a}^T\mathbf{y} - \mathbf{a}^T\mathbf{z} = \beta - \beta = 0$$

we know the normal vector $\mathbf{a}$ is orthogonal to all vectors that are parallel to the hyperplane H. Moreover, for each vector $\mathbf{z}$ in H and $\mathbf{w}$ in H_L^i,

$$\mathbf{a}^T(\mathbf{w} - \mathbf{z}) = \mathbf{a}^T\mathbf{w} - \mathbf{a}^T\mathbf{z} < \beta - \beta = 0$$

This shows that the normal vector $\mathbf{a}$ makes an obtuse angle with any vector that points from the hyperplane toward the interior of H_L. In other words, $\mathbf{a}$ is directed toward the exterior of H_L. Figure 2.1 illustrates this geometry.

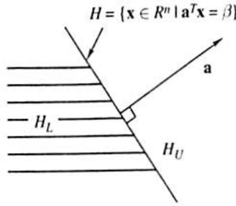

Figure 2.1

For a linear program in its standard form, the hyperplanes

$$H = \{\mathbf{x} \in R^n \mid \mathbf{c}^T\mathbf{x} = \beta\}, \quad \text{for} \quad \beta \in R$$

depict the contours of the linear objective function, and the cost vector $\mathbf{c}$ becomes the normal of its contour hyperplanes.

We further define a *polyhedral set* or *polyhedron* to be a set formed by the intersection of a finite number of closed halfspaces. If the intersection is nonvoid and bounded, it is called a *polytope*. For a linear program in its standard form, if we denote $\mathbf{a}_i$ to be the ith row of the constraint matrix $\mathbf{A}$ and b_i the ith element of the right-hand vector $\mathbf{b}$, then we have m-hyperplanes

$$H_i = \{\mathbf{x} \in R^n \mid \mathbf{a}_i^T\mathbf{x} = b_i\}, \qquad i = 1, \ldots, m$$

and the feasible domain P becomes the intersection of these hyperplanes and the first orthant of R^n. Notice that each hyperplane H is an intersection of two closed halfspaces H_L and H_U and the first orthant of R^n is the intersection of n closed halfspaces $\{\mathbf{x} \in R^n \mid x_i \geq 0\}$ $(i = 1, 2, \ldots, n)$. Hence the feasible domain P is a polyhedral set. An optimal solution of the linear program can be easily identified if we see how the contour hyperplanes formed by the cost vector $\mathbf{c}$ intersect with the polyhedron formed by the constraints.

Consider the following linear programming problem:

Example 2.1

$$\begin{aligned} \text{Minimize} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 40 \\ & 2x_1 + x_2 + x_4 = 60 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Although it has four variables, the feasible domain can be represented as a two-dimensional graph defined by

$$x_1 + x_2 \leq 40, \qquad 2x_1 + x_2 \leq 60, \qquad x_1 \geq 0, \qquad x_2 \geq 0$$

Hence we see a graphical representation in Figure 2.2.

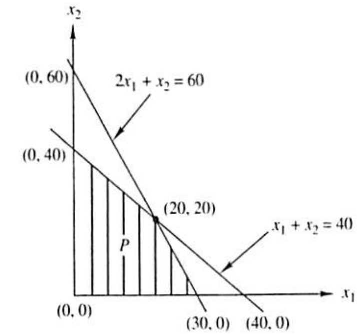

Figure 2.2

2.3 AFFINE SETS, CONVEX SETS, AND CONES

A more detailed study of polyhedral sets and polytopes requires the following definition:

Given p points, $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p \in R^n$, and p scalars $\lambda_1, \lambda_2, \ldots, \lambda_p \in R$, the expression $\lambda_1\mathbf{x}^1 + \lambda_2\mathbf{x}^2 + \cdots + \lambda_p\mathbf{x}^p$ is called a *linear combination*. The linear combination becomes an *affine combination* when $\lambda_1 + \lambda_2 + \cdots + \lambda_p = 1$; a *convex combination* when $\lambda_1 + \lambda_2 + \cdots + \lambda_p = 1$ and $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_p \leq 1$; and a *convex conical combination* when $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_p$.

To understand the geometrical meaning of the affine and convex combinations, we consider the case of two points $\mathbf{x}^1$ and $\mathbf{x}^2$ and its linear combination. Since we can always let $\lambda_1 = 1 - s$ and $\lambda_2 = s$, for a scalar s to replace the equation $\lambda_1 + \lambda_2 = 1$, we see that

$$\lambda_1\mathbf{x}^1 + \lambda_2\mathbf{x}^2 = (1 - s)\mathbf{x}^1 + s\mathbf{x}^2 = \mathbf{x}^1 + s(\mathbf{x}^2 - \mathbf{x}^1)$$

Consequently, we know the set of all affine combinations of distinct points $\mathbf{x}^1, \mathbf{x}^2 \in R^n$ is the whole line determined by these two points, while the set of all convex combinations is the line segment jointing $\mathbf{x}^1$ and $\mathbf{x}^2$. Obviously each convex combination is an affine combination, but the converse statement holds only when $\mathbf{x}^1 = \mathbf{x}^2$.

Following the previous definition, for a nonempty subset $S \subset R^n$, we say S is *affine* if S contains every affine combination of any two points $\mathbf{x}^1, \mathbf{x}^2 \in S$; S is *convex* if S contains every convex combination of any two points $\mathbf{x}^1, \mathbf{x}^2 \in S$.

It is clear that affine sets are convex, but convex sets need not be affine. Moreover, the intersection of a collection (either finite or infinite) of affine sets is either empty or affine and the intersection of a collection (either finite or infinite) of convex sets is either empty or convex.

We may notice that hyperplanes are affine (and hence convex), but closed halfspaces are convex only (not affine). Hence the *linear manifold* (the solution set of a finite system of linear equations) $\{\mathbf{x} \in R^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is affine (and hence convex) but the feasible domain P of our linear program is convex only.

Given a set $S \subset R^n$ and $\mathbf{x} \in S$, we say $\mathbf{x}$ is an interior point of S, if there exists a scalar $\epsilon > 0$ such that the open ball $B = \{\mathbf{y} \in R^n \mid ||\mathbf{x} - \mathbf{y}|| < \epsilon\}$ is contained in S. Otherwise $\mathbf{x}$ is a *boundary point* of S.

For a convex set $S \subset R^n$, a key geometric property is due to the following separation theorem:

Separation Theorem. Let S be a convex subset of R^n and $\mathbf{x}$ be a boundary point of S. Then there is a hyperplane H containing $\mathbf{x}$ with S contained in either H_L or H_U.

Based on this theorem, we can define a *supporting hyperplane* H to be a hyperplane such that (i) the intersection of H and S is not empty, and (ii) H_L contains S. A picture of a supporting hyperplane to a convex set is given by Figure 2.3.

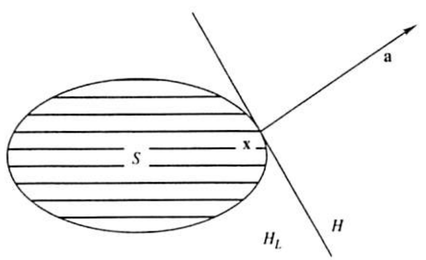

Figure 2.3

One very important fact to point out here is that the intersection set of the polyhedral set P and the supporting hyperplane with the negative cost vector $-\mathbf{c}$ as its normal provides optimal solutions to our linear programming problem. This fact will be proved in Exercise 1.6, and this is the key idea of solving linear programming problems by "graphic method." Figure 2.4 illustrates this situation of Example 1.

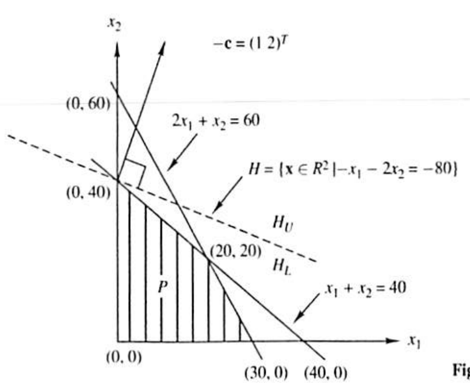

Figure 2.4

In general, for a convex polyhedral set P and a supporting hyperplane H, the intersection set $F = P \cap H$ is called a *face* of P. If F is a zero-dimensional set, we have a *vertex;* one-dimensional, an *edge;* and one dimension less than set P, a *facet.* To define the dimensionality of a subset of R^n, we start with an affine subspace. For a subspace $S \subset R^n$ and a vector $\mathbf{a} \in R^n$, the set

$$S_a = \{\mathbf{y} = \mathbf{x} + \mathbf{a} \mid \mathbf{x} \in S\} \tag{2.7}$$

is called an *affine subspace* of R^n. Basically, translating a subspace by a vector results in an affine subspace. The *dimension* of S_a is equal to the maximum number of linearly independent vectors in S. The *dimension* of a subset $C \subset R^n$ is then defined to be the smallest dimension of any affine subspace which contains C.

One more important structure to define is the conical set. A nonempty set $C \subset R^n$ is a *cone* if $\lambda\mathbf{x} \in C$ for each $\mathbf{x} \in C$ and $\lambda \geq 0$. It is obvious that each cone contains the zero vector. Moreover, a cone that contains at least one nonzero vector $\mathbf{x}$ must contain the "ray" of $\mathbf{x}$, namely $\{\lambda\mathbf{x} \mid \lambda \geq 0\}$. Such cones can clearly be viewed as the union of rays. A cone needs not to be convex, but given an $m \times n$ matrix $\mathbf{M}$, a convex cone can be generated by the columns of $\mathbf{M}$, namely

$$\mathbf{M}_c = \{\mathbf{y} \in R^m \mid \mathbf{y} = \mathbf{M}\mathbf{w}, \mathbf{w} \in R^n, \mathbf{w} \geq \mathbf{0}\} \tag{2.8}$$

This particular cone will be used in later chapters.

Affine sets, convex sets, and convex cones have certain important properties in common. Given a nonempty set $S \subset R^n$, the set of all affine (convex, convex conical) combinations of points in S is an affine (convex, convex conical) set which is identical to the intersection of all affine (convex, convex conical) sets containing S. We called this set an *affine (convex, convex conical correspondingly) hull.*

2.4 EXTREME POINTS AND BASIC FEASIBLE SOLUTIONS

Extreme points of a polyhedral set are geometric entities, while the basic feasible solutions of a system of linear equations and inequalities are defined algebraically. When these two basic concepts are linked together, we have algebraic tools, guided by geometric intuition, to solve linear programming problems.

The definition of extreme points is stated here: A point $\mathbf{x}$ in a convex set C is said to be an *extreme point* of C if $\mathbf{x}$ is not a convex combination of any other two distinct points in C. In other words, an extreme point is a point that does not lie strictly within the line segment connecting two other points of the convex set. From the pictures of convex polyhedral sets, especially in lower-dimensional spaces, it is clear to see that the extreme points are those "vertices" of a convex polyhedron. A formal proof is left as an exercise.

To characterize those extreme points of the feasible domain $P = \{\mathbf{x} \in R^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ of a given linear program in its standard form, we may assume that $\mathbf{A}$ is an $m \times n$ matrix with $m \leq n$. We also denote the jth column of $\mathbf{A}$ by $\mathbf{A}_j$, for $j = 1, 2, \ldots, n$.

Then for each point $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in P$, we have

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n = \mathbf{b} \tag{2.9}$$

Therefore we call column $\mathbf{A}_j$ the corresponding column of the jth component x_j of $\mathbf{x}$, for $j = 1, \ldots, n$. Moreover, we have the following theorem.

Theorem 2.1. A point $\mathbf{x}$ of the polyhedral set $P = \{\mathbf{x} \in R^n \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is an extreme point of P if and only if the columns of $\mathbf{A}$ corresponding to the positive components of $\mathbf{x}$ are linearly independent.

Proof. Without loss of generality, we may assume that the components of $\mathbf{x}$ are zero except for the first p components, namely

$$\mathbf{x} = \begin{pmatrix} \bar{\mathbf{x}} \\ \hline \mathbf{0} \end{pmatrix} \quad \text{where} \quad \bar{\mathbf{x}} = (x_1, \ldots, x_p)^T > \mathbf{0}$$

We also denote the first p columns of matrix $\mathbf{A}$ by $\bar{\mathbf{A}}$. Hence $\mathbf{Ax} = \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{b}$.

($\geq$ side): Suppose that the columns of $\bar{\mathbf{A}}$ are not linearly independent, then there exists a nonzero vector $\bar{\mathbf{w}}$ such that $\bar{\mathbf{A}}\bar{\mathbf{w}} = \mathbf{0}$. We define $\bar{\mathbf{y}}^1 = \bar{\mathbf{x}} + \delta\bar{\mathbf{w}}$ and $\bar{\mathbf{y}}^2 = \bar{\mathbf{x}} - \delta\bar{\mathbf{w}}$. For a small enough $\delta > 0$, we see $\bar{\mathbf{y}}^1, \bar{\mathbf{y}}^2 \geq \mathbf{0}$, and $\bar{\mathbf{A}}\bar{\mathbf{y}}^1 = \bar{\mathbf{A}}\bar{\mathbf{y}}^2 = \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{b}$. We further define

$$\mathbf{y}^1 = \begin{pmatrix} \bar{\mathbf{y}}^1 \\ \hline \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{y}^2 = \begin{pmatrix} \bar{\mathbf{y}}^2 \\ \hline \mathbf{0} \end{pmatrix}$$

Then we know $\mathbf{y}^1, \mathbf{y}^2 \in P$ and $\mathbf{x} = 1/2\mathbf{y}^1 + 1/2\mathbf{y}^2$. In other words, $\mathbf{x}$ is not an extreme point of P.

($\leq$ side): Suppose that $\mathbf{x}$ is not an extreme point, then $\mathbf{x} = \lambda\mathbf{y}^1 + (1 - \lambda)\mathbf{y}^2$ for some distinct $\mathbf{y}^1, \mathbf{y}^2 \in P$ and $0 \leq \lambda \leq 1$. Since $\mathbf{y}^1, \mathbf{y}^2 \geq 0$ and $0 \leq \lambda \leq 1$, the last $n - p$ components of $\mathbf{y}^1$ must be zero. Consequently, we have a nonzero vector $\mathbf{w} = \mathbf{x} - \mathbf{y}^1$ such that $\bar{\mathbf{A}}\mathbf{w} = \mathbf{Aw} = \mathbf{Ax} - \mathbf{Ay}^1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. This shows that the columns of $\bar{\mathbf{A}}$ are linearly dependent.

For an $m \times n$ matrix $\mathbf{A}$ (assuming $m \leq n$), if there exist m linearly independent columns of $\mathbf{A}$, we say $\mathbf{A}$ has *full row rank*, or *full rank* in short. In this case, we can group those m linearly independent columns together to form a *basis* $\mathbf{B}$ and leave the remaining $n - m$ columns as *nonbasis* $\mathbf{N}$. In other words, we can rearrange $\mathbf{A} = [\mathbf{B} \mid \mathbf{N}]$. We also rearrange the components of any solution vector $\mathbf{x}$ in the corresponding order, namely

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \hline \mathbf{x}_N \end{pmatrix}$$

For a component in $\mathbf{x}_B$, its corresponding columns is in the basis $\mathbf{B}$, we call it a *basic variable*. Similarly, those components in $\mathbf{x}_N$ are called *nonbasic variables*. Since $\mathbf{B}$ is a nonsingular $m \times m$ matrix, we can always set all nonbasic variables to be zero, i.e.,

$\mathbf{x}_N = \mathbf{0}$, and solve the system of equations $\mathbf{Bx}_B = \mathbf{b}$ for basic variables. Then vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \hline \mathbf{x}_N \end{bmatrix}$$

becomes a *basic solution*. Furthermore, if $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, then we say $\mathbf{x}$ is a *basic feasible solution* to the linear program.

If matrix $\mathbf{A}$ does not have full row rank, then either the system of equations $\mathbf{Ax} = \mathbf{b}$ has no solution (hence $P = \phi$) or some constraints are redundant. After removing redundant constraints from $\mathbf{A}$, the remaining matrix has full row rank. Therefore, we assume that the constraint matrix $\mathbf{A}$ of a given linear programming problem always have full row rank unless specified otherwise. Under this assumption, since there are at most

$$C(n, m) = \frac{n!}{m!(n - m)!}$$

different ways of choosing m linearly independent columns from n columns of $\mathbf{A}$, we know there are at most $C(n, m)$ basic solutions.

The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.1.1. A point $\mathbf{x} \in P = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is an extreme point of P if and only if $\mathbf{x}$ is a basic feasible solution corresponding to some basis $\mathbf{B}$.

By noticing that every basic feasible solution is a basic solution, we have the next corollary.

Corollary 2.1.2. For a given linear program in its standard form, there are at most $C(n, m)$ extreme points in its feasible domain P.

2.5 NONDEGENERACY AND ADJACENCY

A very important fact to mention is that the correspondence between basic feasible solutions and extreme points of P, as described in Corollary 2.2, in general is not one-to-one. Corresponding to each basic feasible solution there is a unique extreme point in P, but corresponding to each extreme point in P there may be more than one basic feasible solution.

Consider the a polytope P define by

$$P = \{\mathbf{x} \in R^4 \mid x_1 + x_2 + x_3 = 10, x_1 + x_4 = 10, x_1, x_2, x_3, x_4 \geq 0\} \tag{2.10}$$

or, equivalently for its graph in Figure 2.5, we have

$$P = \{\mathbf{x} \in R^2 \mid x_1 + x_2 \leq 10, x_1 \leq 10, x_1, x_2 \geq 0\} \tag{2.11}$$

Note that P has three extreme points in Figure 2.5, namely,

$$A = (0, 0), \qquad B = (0, 10), \qquad \text{and} \qquad C = (10, 0)$$

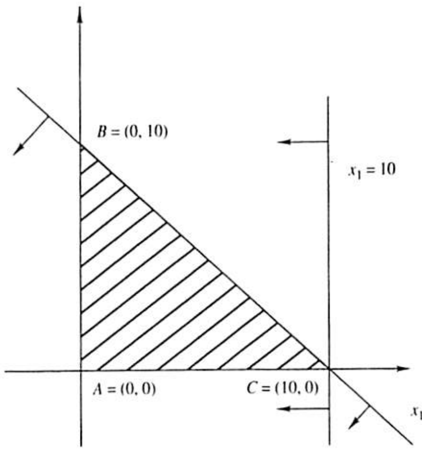

Figure 2.5

Their four-dimensional coordinates corresponding to (2.10) are

$$A = (0, 0, 10, 10), \quad B = (0, 10, 0, 10), \quad \text{and} \quad C = (10, 0, 0, 0)$$

We can check that extreme point A corresponds to one basic feasible solution taking x_3 and x_4 as basic variables, and extreme point B corresponds to the basic feasible solution taking x_2 and x_4 as basic variables. However, extreme point C corresponds to three basic feasible solutions: one takes x_1 and x_2 as basic variables, one takes x_1 and x_3, and the remaining one takes x_1 and x_4. The reason for C getting more than one corresponding basic feasible solution is that all the three corresponding basic feasible solutions have one basic variable with zero in value, which makes them indistinguishable from one another. Based on this observation, we define a basic feasible solution to be *nondegenerate*, if it has exactly m positive basic variables. Otherwise, the basic feasible solution has at least $n - m + 1$ zero elements in it, and we call it a *degenerate* case.

A linear programming problem is *nondegenerate* if all basic feasible solutions are nondegenerate. In this case, there is a one-to-one correspondence between the extreme points and basic feasible solutions. This *nondegeneracy assumption* of a given linear programming problem will greatly simplify our situations in solving linear programming problems. We shall discuss it further in the next chapter.

Two basic feasible solutions of P are *adjacent*, if they use $m - 1$ basic variables in common to form basis. For example, in Figure 2.2, it is easy to verify that extreme point (0, 40) is adjacent to (0, 0) but not adjacent to (30, 0) since (0, 40) takes x_2 and x_4 as basic variables, while (0, 0) takes x_3 and x_4 and (30, 0) takes x_1 and x_3. Under the "nondegeneracy assumption," since each of the $n - m$ nonbasic variables could replace one current basic variable in a given basic feasible solution, we know that every basic feasible solution (hence its corresponding extreme point) has exactly $n - m$ adjacent neighbors. Actually, each adjacent basic feasible solution can be reached by increasing

the value of one nonbasic variable from zero to positive and decreasing the value of one basic variable from positive to zero. This is the basic concept of *pivoting* in the simplex method to be studied in the next chapter. Geometrically, adjacent extreme points of P are linked together by an *edge* of P, and pivoting leads one to move from one extreme point to its adjacent neighbor along the edge direction. This can be clearly seen in Figure 2.2.

2.6 RESOLUTION THEOREM FOR CONVEX POLYHEDRONS

Suppose the feasible domain P is bounded—in other words, P is a polytope. From Figure 2.6 it is easy to observe that each point of P can be represented as a convex combination of the finite number of extreme points of P.

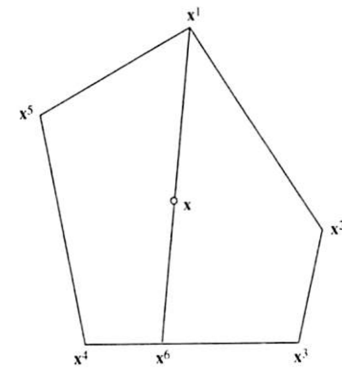

Figure 2.6

This idea of "convex resolution" can be verified for a general polyhedral set with the help of the following definition: An *extremal direction* of a polyhedral set P is a nonzero vector $\mathbf{d} \in R^n$ such that for each $\mathbf{x}^0 \in P$ the ray $\{\mathbf{x} \in R^n \mid \mathbf{x} = \mathbf{x}^0 + \lambda\mathbf{d}, \lambda \geq 0\}$ is contained in P. Note that, in the convex analysis literature, it is usually called the *direction of recession*.

From the definition of the feasible domain P, we see that a nonzero vector $\mathbf{d} \in R^n$ is an extremal direction of P if and only if $\mathbf{Ad} = \mathbf{0}$ and $\mathbf{d} \geq \mathbf{0}$. Also, P is unbounded if and only if P has an extremal direction. Using extreme points and extremal directions, every point in P can be well represented by the following theorem.

Theorem 2.2 (Resolution Theorem). Let $V = \{\mathbf{v}^i \in R^n \mid i \in I\}$ be the set of all extreme points of P with a finite index set I. Then for each $\mathbf{x} \in P$, we have

$$\mathbf{x} = \sum_{i \in I} \lambda_i \mathbf{v}^i + \mathbf{d} \tag{2.9}$$

where $\sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0$ for $i \in I$, and $\mathbf{d}$ is either the zero vector or an extremal direction of P.

A proof using the mathematical induction method on the number of positive components of the given vector $\mathbf{x} \in P$ is included at the end of this chapter as an exercise.

A direct consequence of the resolution theorem confirms our observation made at the beginning of this section, namely,

Corollary 2.2.1. If P is bounded (a polytope), then each point $\mathbf{x} \in P$ is a convex combination of the extreme points of P.

Another direct implication is as follows.

Corollary 2.2.2. If P is nonempty, then it has at least one extreme point.

2.7 FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

The resolution theorem reveals one fundamental property of linear programming for algorithm design.

Theorem 2.3 (Fundamental Theorem of Linear Programming). For a consistent linear program in its standard form with a feasible domain P, the minimum objective value of $\mathbf{z} = \mathbf{c}^T\mathbf{x}$ over P is either unbounded below or is achievable at least at one extreme point of P.

Proof. Let $V = \{\mathbf{v}^i \in P \mid i \in I\}$ be the set of all extreme points of P with a finite index set I. Since the problem is consistent, I is nonempty and there is at least one $\mathbf{v}^1 \in V$. By the resolution theorem, P either has an external direction $\mathbf{d}$ with $\mathbf{c}^T\mathbf{d} < 0$ or does not have such a direction.

In the first case, P is unbounded, and $\mathbf{z}$ goes to minus infinity at $\mathbf{v}^1 + \lambda\mathbf{d}$ as λ goes to positive infinity.

For the latter, for each $\mathbf{x} \in P$, either

$$\mathbf{x} = \sum_{i \in I} \lambda_i \mathbf{v}^i \quad \text{with} \quad \sum_{i \in I} \lambda_i = 1, \quad \lambda_i \geq 0 \quad \text{or}$$

$$\mathbf{x} = \sum_{i \in I} \lambda_i \mathbf{v}^i + \bar{\mathbf{d}} \quad \text{with} \quad \sum_{i \in I} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \text{and} \quad \mathbf{c}^T\bar{\mathbf{d}} \geq 0$$

In both situations, assuming $\mathbf{c}^T\mathbf{v}^{\min}$ is the minimum among $\{\mathbf{c}^T\mathbf{v}^i \mid i \in I\}$, we have

$$\mathbf{c}^T\mathbf{x} \geq \sum_{i \in I} \lambda_i (\mathbf{c}^T\mathbf{v}^i) \geq \mathbf{c}^T\mathbf{v}^{\min}\left(\sum_{i \in I} \lambda_i\right) = \mathbf{c}^T\mathbf{v}^{\min}$$

Hence the minimum value of z is attained at the extreme point $\mathbf{v}^{\min}$.

It is important to point out that Theorem 2.3 does not rule out the possibility of having an optimal solution at a nonextreme point. It simply says that among all the optimal solutions to a given linear programming problem, at least one of them is an extreme point.

2.8 CONCLUDING REMARKS: MOTIVATIONS OF DIFFERENT APPROACHES

The fundamental theorem of linear programming shows that one of the extreme points of the feasible domain P is an optimal solution to a consistent linear programming problem unless the problem is unbounded. This fundamental property has guided the design of algorithms for linear programming.

One of the most intuitive ways of solving a linear programming problem is the *graphical method*, as we discussed before. We draw a graph of the feasible domain P first. Then at each extreme point $\mathbf{v}$ of P, using the negative cost vector $-\mathbf{c}$ as the normal vector, we draw a hyperplane H. If P is contained in the halfspace H_L, then H is a desired supporting hyperplane and $\mathbf{v}$ is an optimal solution to the given linear programming problem. This method provides us a clear picture, but it is limited to those problems whose feasible domains can be drawn in the three-dimensional, or lower, spaces only.

Another straightforward method is the *enumeration method*. Since an extreme point corresponds to a basic feasible solution, it must be a basic solution. We can generate all basic solutions by choosing m linearly independent columns from the columns of constraint matrix $\mathbf{A}$ and solving the corresponding system of linear equations. Among all basic solutions, we identify feasible ones and take the optimal one as our solution. The deficiency of this method is due to the laborious computation. It becomes impractical when the number $C(n, m)$ becomes large.

The rest of this book is devoted to designing efficient iterative algorithms for linear programming. There are two basic approaches. One is the well-known simplex method, the other is the newly developed interior-point approach. Focusing on finding an optimal extreme point, the simplex approach starts with one extreme point, hops to a better neighboring extreme point along the boundary, and finally stops at an optimal extreme point. Because the method is well designed, rarely do we have to visit too many extreme points before an optimal one is found. But, in the worst case, this method may still visit all nonoptimal extreme points.

Unlike the simplex method, the interior-point method stays in the interior of P and tries to position a current solution as the "center of universe" in finding a better direction for the next move. By properly choosing step lengths, an optimal solution is finally achieved after a number of iterations. This approach takes more effort, hence more computational time, in finding a moving direction than the simplex method, but better moving directions result in fewer iterations. Therefore the interior-point approach has become a rival of the simplex method and gathered much attention.

Figure 2.7 shows the fundamental difference between these two approaches.

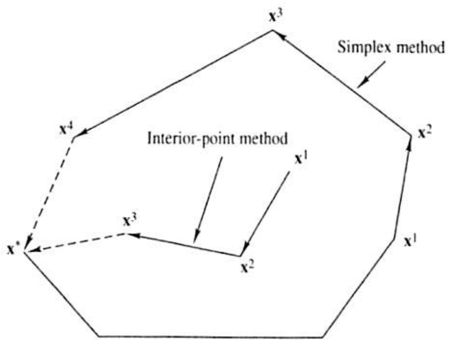

Figure 2.7

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EXERCISES

2.1. Prove that a linear program with bounded, feasible domain must be bounded, and give a counterexample to show that the converse statement need not be true.

2.2. Let S be a subset of R^n. For each of the following assertions, either prove it or provide a counterexample in R^2 to disprove it:

(a) If S is convex, then S is (i) affine; (ii) a cone; (iii) a polyhedron; (iv) a polytope.

(b) If S is affine, then S is (i) convex; (ii) a cone; (iii) a polyhedron; (iv) a polytope.

(c) If S is a cone, then S is (i) convex; (ii) affine; (iii) a polyhedron; (iv) a polytope.

(d) If S is a polyhedron, then S is (i) convex; (ii) affine; (iii) a cone; (iv) a polytope.

(e) If S is a polytope, then S is (i) convex; (ii) affine; (iii) a cone; (iv) a polyhedron.

2.3. Let $H = \{\mathbf{x} \in R^n \mid \mathbf{a}^T\mathbf{x} = \beta\}$ be a hyperplane. Show that H is affine and convex.

2.4. Suppose $C_1, C_2, \ldots, C_p$ are $p(>0)$ convex subsets of R^n. Prove or disprove the following assertions:

(a) $\bigcap_{i=1}^{p} C_i$ is convex.

(b) $\bigcup_{i=1}^{p} C_i$ is convex.

2.5. Use the results of Exercises 2.3 and 2.4 to show that $P = \{\mathbf{x} \in R^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is a convex polyhedron.

2.6. To make the graphic method work, prove that the intersection set of the feasible domain P and the supporting hyperplane whose normal is given by the negative cost vector $-\mathbf{c}$ provides the optimal solutions to a given linear programming problem.

2.7. Let $P = \{(x_1, x_2) \in R^2 \mid x_1 + x_2 \leq 40, 2x_1 + x_2 \leq 60, x_1 \leq 20, x_1, x_2 \geq 0\}$. Do the following:

(a) Draw the graph of P.

(b) Convert P to the standard equality form.

(c) Generate all basic solutions.

(d) Find all basic feasible solutions.

(e) For each basic feasible solution, point out its corresponding extreme points in the graph of P.

(f) Which extreme points correspond to degenerate basic feasible solutions?

2.8. For P as defined in Exercise 2.7, use the graphic method to solve linear programming problems with the following objective functions:

(a) $z = -x_2$;

(b) $z = -x_1 - x_2$;

(c) $z = -2x_1 - x_2$;

(d) $z = -x_1$

(e) $z = -x_1 + x_2$.

What conclusion can be reached on the optimal solution set P^*?

2.9. Show that the set of all optimal solutions to a linear programming problem is a convex set. Now, can you construct a linear programming problem which has exactly two different optimal solutions? Why?

2.10. Prove that for a degenerate basic feasible solution with $p < m$ positive elements, its corresponding extreme point P may correspond to $C(n-p, n-m)$ different basic feasible solutions at the same time.

2.11. Let $\mathbf{M}$ be the 2×2 identity matrix. Show that

(a) $\mathbf{M}_c$, the convex cone generated by $\mathbf{M}$, is the first orthant of R^2.

(b) $\mathbf{M}_c$ is the smallest convex cone that which contains the column vectors $(1, 0)^T$ and $(0, 1)^T$.

2.12. Given a nonempty set $S \subset R^n$, show that the set of all affine (convex, convex conical) combinations of points in S is an affine (convex, convex conical) set which is identical to the intersection of all affine (convex, convex conical) sets containing S.