(c) Minimize $-7x_5 - 4x_6 + 15x_7$

subject to

$$\begin{aligned} x_1 + (1/3)x_5 - (32/9)x_6 + (20/9)x_7 &= 0 \\ x_2 + (1/6)x_5 - (13/9)x_6 + (5/18)x_7 &= 0 \\ x_3 + (2/3)x_5 - (16/9)x_6 + (1/9)x_7 &= 0 \\ x_4 &= 1 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

(d) Minimize $3x_1 - 2x_2 + 4x_3 - x_4 + x_5$

subject to

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 + 4x_5 &= 12 \\ 4x_1 - 5x_2 + 3x_3 - 4x_5 &= 10 \\ 3x_1 - x_2 + 2x_3 + 2x_4 &= 8 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

(e) Minimize $-x_3$

subject to

$$\begin{aligned} x_1 &\leq 1 \\ x_2 &\geq 0.00000001x_1 \\ x_2 &\leq 1 - 0.00000001x_1 \\ x_3 &\geq 0.00000001x_2 \\ x_3 &\leq 1 - 0.00000001x_2 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

3.17. Analyze the computer outputs of 3.16 and comment on the special properties of each subproblem.

4

Duality Theory and Sensitivity Analysis

The notion of duality is one of the most important concepts in linear programming. Basically, associated with each linear programming problem (we may call it the *primal problem*), defined by the constraint matrix **A**, the right-hand-side vector **b**, and the cost vector **c**, there is a corresponding linear programming problem (called the *dual problem*) which is constructed by the same set of data **A**, **b**, and **c**. A pair of primal and dual problems are closely related. The interesting relationship between the primal and dual reveals important insights into solving linear programming problems.

To begin this chapter, we introduce a dual problem for the standard-form linear programming problem. Then we study the fundamental relationship between the primal and dual problems. Both the "strong" and "weak" duality theorems will be presented. An economic interpretation of the dual variables and dual problem further exploits the concepts in duality theory. These concepts are then used to derive two important simplex algorithms, namely the *dual simplex algorithm* and the *primal dual algorithm*, for solving linear programming problems.

We conclude this chapter with the *sensitivity analysis*, which is the study of the effects of changes in the parameters (**A**, **b**, and **c**) of a linear programming problem on its optimal solution. In particular, we study different methods of changing the cost vector, changing the right-hand-side vector, adding and removing a variable, and adding and removing a constraint in linear programming.

4.1 DUAL LINEAR PROGRAM

Consider a linear programming problem in its standard form:

$$\text{Minimize } \mathbf{c}^T\mathbf{x} \tag{4.1a}$$

$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b} \tag{4.1b}$$

$$\mathbf{x} \geq \mathbf{0} \tag{4.1c}$$

where $\mathbf{c}$ and $\mathbf{x}$ are n-dimensional column vectors, $\mathbf{A}$ an $m \times n$ matrix, and $\mathbf{b}$ an m-dimensional column vector. Let us assume that $\mathbf{x}$ is a nondegenerate basic feasible solution. Corresponding to $\mathbf{x}$, we have a basis matrix $\mathbf{B}$ and nonbasis matrix $\mathbf{N}$. Also let $\tilde{\mathbf{B}}$ be the index set of basic variables and $\tilde{\mathbf{N}}$ the index set of nonbasic variables.

According to Theorem 3.1, we know $\mathbf{x}$ is optimal if and only if $r_q \geq 0$, for each $q \in \tilde{\mathbf{N}}$, or, equivalently, $\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_q \leq c_q$ for each $q \in \tilde{\mathbf{N}}$. Also it is easy to see that $\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_p = c_p$ for each $p \in \tilde{\mathbf{B}}$. Therefore, by denoting $\mathbf{c}_B^T\mathbf{B}^{-1} = \mathbf{w}^T$, we have

$$\mathbf{w}^T\mathbf{A}_q \leq c_q \qquad \forall q \in \tilde{\mathbf{N}}$$

and

$$\mathbf{w}^T\mathbf{A}_p = c_p \qquad \forall p \in \tilde{\mathbf{B}}$$

Therefore, in vector form, we have

$$\mathbf{w}^T[\mathbf{B} \mid \mathbf{N}] \leq \mathbf{c}^T$$

or, equivalently,

$$\mathbf{A}^T\mathbf{w} \leq \mathbf{c} \tag{4.2}$$

Notice that, at this optimal solution $\mathbf{x}$, since $\mathbf{x}_N = \mathbf{0}$, we see

$$\mathbf{b}^T\mathbf{w} = \mathbf{w}^T\mathbf{b} = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}^T\mathbf{x}$$

But in general we only have

$$\mathbf{b}^T\mathbf{w} = \mathbf{w}^T\mathbf{b} = \mathbf{w}^T\mathbf{A}\mathbf{x} \leq \mathbf{c}^T\mathbf{x}, \quad \text{for} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}$$

Therefore, in view of the new variables $\mathbf{w}$, we can define an associate linear programming problem:

$$\text{Maximize } \mathbf{b}^T\mathbf{w} \tag{4.3a}$$

$$\text{subject to } \mathbf{A}^T\mathbf{w} \leq \mathbf{c};\ \mathbf{w} \text{ unrestricted} \tag{4.3b}$$

Notice that problem (4.3) is a maximization problem with m unrestricted variables and n inequality constraints. The roles of the variables and constraints are somewhat reversed in problems (4.1) and (4.3). Usually, we call problem (4.1) the *primal problem* and problem (4.3) the *dual problem.* These two make a *primal-dual pair.* A vector $\mathbf{w}$ of dual variables becomes a *dual solution,* if constraint (4.3b) is satisfied.

Example 4.1

The dual linear program of Example 2.1 becomes

$$\begin{array}{lll} \text{Maximize} & 40w_1 + 60w_2 & \\ \text{subject to} & w_1 + 2w_2 & \leq -1 \\ & w_1 + w_2 & \leq -2 \\ & w_1, w_2 & \leq 0 \end{array}$$

Example 4.2

For a linear programming problem in the "inequality form," i.e.,

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } \mathbf{A}\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$$

We can convert this problem into its standard form and then derive its dual problem. As we are required to show in an exercise, it is the following:

$$\text{Maximize } \mathbf{b}^T\mathbf{w}$$

$$\text{subject to } \mathbf{A}^T\mathbf{w} \leq \mathbf{c}, \quad \mathbf{w} \geq \mathbf{0}$$

These two linear programming problems are sometimes called a *symmetric pair* of primal and dual programs, owing to the symmetric structure observed.

4.2 DUALITY THEORY

Note that both the primal and dual problems are defined by the same data set $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. In this section, we study the fundamental relationship between the pair. First we show that the concept of dual problem is well defined in the sense that we can choose either one of the primal-dual pair as the primal problem and the other one becomes its dual problem.

Lemma 4.1. Given a primal linear program, the dual problem of the dual linear program becomes the original primal problem.

Proof. Let us start with problem (4.1). Its dual problem (4.3) may be expressed as

$$-\text{Minimize } z = -\mathbf{b}^T\mathbf{w} \tag{4.4a}$$

$$\text{subject to } \mathbf{A}^T\mathbf{w} \leq \mathbf{c}; \quad \mathbf{w} \text{ unrestricted} \tag{4.4b}$$

Since $\mathbf{w}$ is unrestricted, we may represent $\mathbf{w} = \mathbf{u} - \mathbf{v}$ with $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$. To convert problem (4.4) into its standard form, we further introduce slack variables $\mathbf{s} \geq \mathbf{0}$

for an equivalent form

$$-\text{Minimize} \quad z = [-\mathbf{b}^T \mid \mathbf{b}^T \mid \mathbf{0}]\mathbf{x} \tag{4.5a}$$

$$\text{subject to} \quad \left[\mathbf{A}^T \mid -\mathbf{A}^T \mid \mathbf{I}\right] \mathbf{x} = \mathbf{c}; \quad \mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \hline \mathbf{v} \\ \hline \mathbf{s} \end{bmatrix} \geq \mathbf{0} \tag{4.5b}$$

Note that problem (4.5) is in its standard form. Its dual problem becomes

$$-\text{Maximize} \quad z = \mathbf{c}^T \mathbf{w} \tag{4.6a}$$

$$\text{subject to} \quad \begin{bmatrix} \mathbf{A} \\ \hline \mathbf{-A} \\ \hline \mathbf{I} \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} -\mathbf{b} \\ \hline \mathbf{b} \\ \hline \mathbf{0} \end{bmatrix}, \quad \mathbf{w} \text{ unrestricted} \tag{4.6b}$$

Defining $\mathbf{x} = -\mathbf{w}$, we have an equivalent problem

$$\text{Minimize} \quad z = \mathbf{c}^T \mathbf{x} \tag{4.7a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \tag{4.7b}$$

which is nothing but the primal linear programming problem (4.1).

Next we show that the primal (minimization) problem is always bounded below by the dual (maximization) problem and the dual (maximization) problem is always bounded above by the primal (minimization) problem, if they are feasible.

Theorem 4.1 (Weak Duality Theorem of LP). If $\mathbf{x}^0$ is a primal feasible solution and $\mathbf{w}^0$ is dual feasible, then $\mathbf{c}^T\mathbf{x}^0 \geq \mathbf{b}^T\mathbf{w}^0$.

Proof. The dual feasibility of $\mathbf{w}^0$ implies that $\mathbf{c} \geq \mathbf{A}^T\mathbf{w}^0$. For $\mathbf{x}^0$ is primal feasible, we know $\mathbf{x}^0 \geq \mathbf{0}$ and, hence, $\mathbf{x}^{0^T}\mathbf{c} \geq \mathbf{x}^{0^T}\mathbf{A}^T\mathbf{w}^0$. Noting also that $\mathbf{A}\mathbf{x}^0 = \mathbf{b}$, we see

$$\mathbf{c}^T\mathbf{x}^0 = \mathbf{x}^{0^T}\mathbf{c} \geq \mathbf{x}^{0^T}\mathbf{A}^T\mathbf{w}^0 = \mathbf{b}^T\mathbf{w}^0.$$

Several corollaries can be immediately obtained from the weak duality theorem:

Corollary 4.1.1. If $\mathbf{x}^0$ is primal feasible, $\mathbf{w}^0$ is dual feasible, and $\mathbf{c}^T\mathbf{x}^0 = \mathbf{b}^T\mathbf{w}^0$, then $\mathbf{x}^0$ and $\mathbf{w}^0$ are optimal solutions to the respective problems.

Proof. Theorem 4.1 indicates that $\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{w}^0 = \mathbf{c}^T\mathbf{x}^0$, for each primal feasible solution $\mathbf{x}$. Thus $\mathbf{x}^0$ is an optimal solution to the primal problem. A similar argument holds for the dual problem.

Corollary 4.1.2. If the primal problem is unbounded below, then the dual problem is infeasible.

Proof. Whenever the dual problem has a feasible solution $\mathbf{w}^0$, the weak duality theorem prevents the primal objective from falling below $\mathbf{b}^T\mathbf{w}^0$.

Similarly, we have the following result:

Corollary 4.1.3. If the dual problem is unbounded above, then the primal problem is infeasible.

Note that the converse statement of either of two foregoing corollaries is not true. For example, when the primal problem is infeasible, the dual could be either unbounded above or infeasible. However, if the primal is infeasible and the dual is feasible, then the dual must be unbounded. Concrete examples are presented in the exercises.

With these results, a stronger result can be stated as the following important theorem.

Theorem 4.2 (strong duality theorem of LP)

1. If either the primal or the dual linear program has a finite optimal solution, then so does the other and they achieve the same optimal objective value.
2. If either problem has an unbounded objective value, then the other has no feasible solution.

Proof. For the first claim, without loss of generality, let us assume that the primal problem has reached a finite optimum at a basic feasible solution $\mathbf{x}$. If we apply the revised simplex method at $\mathbf{x}$ and define $\mathbf{w}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$, then

$$\mathbf{c} - \mathbf{A}^T\mathbf{w} = \begin{bmatrix} \mathbf{c}_\mathrm{B} \\ \hline \mathbf{c}_\mathrm{N} \end{bmatrix} - \begin{bmatrix} \mathbf{B}^\mathrm{T} \\ \hline \mathbf{N}^\mathrm{T} \end{bmatrix} \mathbf{w} = \mathbf{r} \geq \mathbf{0} \tag{4.8}$$

Therefore $\mathbf{w}$ is dual feasible. Moreover, since $\mathbf{x}$ is a basic feasible solution,

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{w}^T\mathbf{b} = \mathbf{b}^T\mathbf{w} \tag{4.9}$$

Owing to Corollary 4.1.1, we know $\mathbf{w}$ is an optimal solution to the dual linear program.

The proof of the second claim is a direct consequence of Corollary 4.1.2 and Corollary 4.1.3.

The strong duality theorem has several implications. First of all, it says there is *no duality gap* between the primal and dual linear programs, i.e., $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{w}^*$. This is not generally true for nonlinear programming problems. Second, in the proof of Theorem 4.2, the *simplex multipliers* (see Section 7 of Chapter 3), or *Lagrange multipliers*, become the vector $\mathbf{w}$ of dual variables. Furthermore, at each iteration of the revised simplex method, the dual vector $\mathbf{w}$ maintains the property $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$. However, unless all components of the reduced costs vector $\mathbf{r}$ are nonnegative, $\mathbf{w}$ is not dual feasible. Thus, the revised simplex method maintains primal feasibility and zero duality gap and seeks for dual feasibility. Needless to say, the simplex multipliers $\mathbf{w}^*$ corresponding to a primal optimal solution $\mathbf{x}^*$ form a dual optimal solution.

A celebrated application of the Duality Theorem is in establishing the existence of solutions to systems of equalities and inequalities. The following result, known as Farka's lemma, concerns this aspect.

Theorem 4.3 (Farka's lemma) The system

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \tag{4.10}$$

has no solution if and only if the system

$$\mathbf{A}^T\mathbf{w} \leq \mathbf{0}, \quad \mathbf{b}^T\mathbf{w} > 0 \tag{4.11}$$

has a solution.

Proof. Consider the (primal) linear program

$$\begin{aligned} &\text{Minimize } \mathbf{0}^T\mathbf{x} \\ &\text{subject to } \mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

and its dual

$$\begin{aligned} &\text{Maximize } \mathbf{b}^T\mathbf{w} \\ &\text{subject to } \mathbf{A}^T\mathbf{w} \leq \mathbf{0}, \quad \mathbf{w} \quad \text{unrestricted} \end{aligned}$$

Since $\mathbf{w} = \mathbf{0}$ is dual feasible, the primal problem is infeasible if and only if the dual is unbounded. However, the primal is infeasible if and only if system (4.10) has no solution; and the dual is unbounded if and only if system (4.11) has a solution. To be more precise, for any solution to system (4.11), say $\mathbf{d}$ such that $\mathbf{A}^T\mathbf{d} \leq \mathbf{0}$ and $\mathbf{b}^T\mathbf{d} > 0$, then $\alpha\mathbf{d}$ is a dual feasible solution that leads to an unbounded objective value as α approaches infinity.

An equivalent statement of Farka's lemma is as follows:
Of the two systems

$$\text{(I)} \quad \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \tag{4.12a}$$

$$\text{(II)} \quad \mathbf{A}^T\mathbf{w} \leq \mathbf{0}, \mathbf{b}^T\mathbf{w} > \mathbf{0} \tag{4.12b}$$

either system (I) or (II) is solvable, but not both.

The geometric implication of this result is quite straightforward. If we denote $\mathbf{A}_j$ as the jth column of $\mathbf{A}$, the existence of a solution to system (4.12a) mandates that $\mathbf{b}$ should lie in the convex cone defined by $\mathbf{A}_j$, for $j = 1, 2, \ldots, n$, since $\mathbf{x} \geq \mathbf{0}$ and

$$\mathbf{b} = \sum_{j=1}^{n} \mathbf{A}_j x_j$$

However, the existence of a solution $\mathbf{w}$ to system (4.12b) requires $\mathbf{w}$ to make an angle greater than 90 degrees with each column of $\mathbf{A}$ while it makes an angle less than 90 degrees with $\mathbf{b}$. Consequently, $\mathbf{b}$ is required to lie outside of the cone defined by the columns of $\mathbf{A}$. Therefore one and only one of the two systems has a solution. Figure 4.1 is a graphic representation of our discussion.

Variants of Farka's lemma, all of them stating that, given a pair of systems of equalities and inequalities, one and only one is solvable, are broadly known as *theorems of the alternative*. We shall introduce some of them in the exercises.

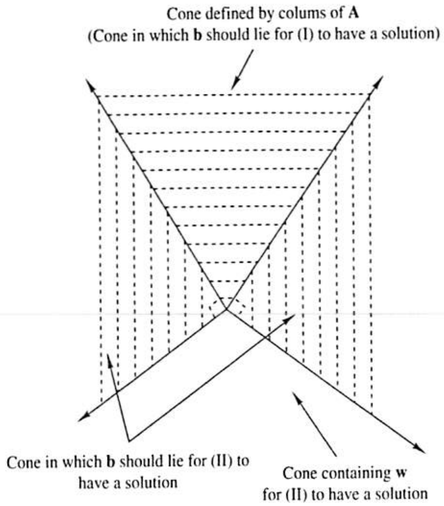

Figure 4.1

Another important application of the duality theory is in establishing optimality conditions for linear programming. In the next section, we first introduce the notion of *complementary slackness* and then study the Karush-Kuhn-Tucker conditions for linear programming problems.

4.3 COMPLEMENTARY SLACKNESS AND OPTIMALITY CONDITIONS

Recall the symmetric pair of primal and dual linear programs:

$$\begin{aligned} &\text{Minimize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } \mathbf{Ax} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{P}$$

$$\begin{aligned} &\text{Maximize } \mathbf{b}^T\mathbf{w} \\ &\text{subject to } \mathbf{A}^T\mathbf{w} \leq \mathbf{c}, \quad \mathbf{w} \geq \mathbf{0} \end{aligned} \tag{D}$$

For the primal problem, we define

$$\mathbf{s} = \mathbf{Ax} - \mathbf{b} \geq \mathbf{0} \tag{4.13}$$

as the *primal slackness vector*. For the dual problem, we define

$$\mathbf{r} = \mathbf{c} - \mathbf{A}^T\mathbf{w} \geq \mathbf{0} \tag{4.14}$$

as the *dual slackness vector*. Notice that $\mathbf{s}$ is an m-dimensional vector and $\mathbf{r}$ an n-dimensional vector. Moreover, for any primal feasible solution $\mathbf{x}$ and dual feasible solution $\mathbf{w}$, we know

$$\begin{aligned} 0 &\le \mathbf{r}^T\mathbf{x} + \mathbf{s}^T\mathbf{w} \\ &= (\mathbf{c}^T - \mathbf{w}^T\mathbf{A})\mathbf{x} + \mathbf{w}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \\ &= \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{w} \end{aligned} \tag{4.15}$$

Therefore, the quantity of $\mathbf{r}^T\mathbf{x} + \mathbf{s}^T\mathbf{w}$ is equal to the duality gap between the primal feasible solution $\mathbf{x}$ and dual feasible solution $\mathbf{w}$. This duality gap vanishes, if, and only if,

$$\mathbf{r}^T\mathbf{x} = 0 \quad \text{and} \quad \mathbf{s}^T\mathbf{w} = 0 \tag{4.16}$$

In this case, $\mathbf{x}$ becomes an optimal primal solution and $\mathbf{w}$ an optimal dual solution. Since all vectors $\mathbf{x}$, $\mathbf{w}$, $\mathbf{r}$, and $\mathbf{s}$ are nonnegative, Equation (4.16) requires that "either $r_j = 0$ or $x_j = 0$ for $j = 1, \ldots, n$" and "either $s_i = 0$ or $w_i = 0$ for $i = 1, \ldots, m$." Hence (4.16) is called the *complementary slackness conditions*. This important result can be summarized as the following theorem:

Theorem 4.4 (Complementary slackness theorem). Let $\mathbf{x}$ be a primal feasible solution and $\mathbf{w}$ be a dual feasible solution to a symmetric pair of linear programs. Then $\mathbf{x}$ and $\mathbf{w}$ become an optimal solution pair if and only if the complementary slackness conditions

$$\begin{aligned} &\text{either} \quad r_j = (\mathbf{c} - \mathbf{A}^T\mathbf{w})_j = 0 \\ &\text{or} \quad x_j = 0, \qquad \forall j = 1, 2, \ldots, n \\ &\text{either} \quad s_i = (\mathbf{A}\mathbf{x} - \mathbf{b})_i = 0 \\ &\text{or} \quad w_i = 0, \qquad \forall i = 1, 2, \ldots, m \end{aligned}$$

are satisfied.

As to the primal-dual pair of linear programs in the standard form, i.e.,

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0} \end{aligned} \tag{P}$$

$$\begin{aligned} &\text{Maximize} \quad \mathbf{b}^T\mathbf{w} \\ &\text{subject to} \quad \mathbf{A}^T\mathbf{w} \le \mathbf{c} \end{aligned} \tag{D}$$

since the primal problem always has zero slackness (they are tight equalities), the condition $\mathbf{w}^T\mathbf{s} = 0$ is automatically met. Therefore, the complementary slackness conditions are simplified to $\mathbf{r}^T\mathbf{x} = 0$.

With this knowledge, we can state the Karush-Kuhn-Tucker (K-K-T) conditions for linear programming problems as following:

Theorem 4.5 (K-K-T optimality conditions for LP). Given a linear programming problem in its standard form, vector $\mathbf{x}$ is an optimal solution to the problem if, and only if, there exist vectors $\mathbf{w}$ and $\mathbf{r}$ such that

(1) $\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0}$ (primal feasibility)
(2) $\mathbf{A}^T\mathbf{w} + \mathbf{r} = \mathbf{c}, \quad \mathbf{r} \ge \mathbf{0}$ (dual feasibility)
(3) $\mathbf{r}^T\mathbf{x} = 0$ (complementary slackness)

In this case, $\mathbf{w}$ is an optimal solution to the dual problem.

Example 4.3

Let us consider Example 3.4. When the revised simplex method terminates, it can be found that $\mathbf{x} = [20 \quad 20 \quad 0 \quad 0]^T$, $\mathbf{w} = [-1 \quad -1]^T$, and $\mathbf{r} = [0 \quad 0 \quad 1 \quad 1]^T$. Hence we know the K-K-T conditions are satisfied, and we have reached an optimal solution.

The theorem of K-K-T conditions is one of the fundamental results in mathematical programming. For a nonlinear programming problem, which is much more general than the linear programming, it specifies the necessary and/or sufficient conditions for optimality, depending upon whether the given problem satisfies certain regularity conditions. A detailed discussion of these regularity conditions is beyond the scope of this book. The result we see in Theorem 4.5 is one special case of the general result.

4.4 AN ECONOMIC INTERPRETATION OF THE DUAL PROBLEM

So far, we have seen that the dual linear program uses the same set of data as the primal problem, supports the primal solutions as a lower bound, and provides insights into the sufficient and necessary conditions for optimality. In this section, we intend to explain the meaning of dual variables and make an economic interpretation of the dual problem.

4.4.1 Dual Variables and Shadow Prices

Given a linear programming problem in its standard form, the primal problem can be viewed as a process of providing different services ($\mathbf{x} \ge \mathbf{0}$) to meet a set of customer demands ($\mathbf{A}\mathbf{x} = \mathbf{b}$) in a least expensive manner with a minimum cost (min $\mathbf{c}^T\mathbf{x}$).

For a nondegenerated optimal solution $\mathbf{x}^*$ obtained by the revised simplex method, we have

$$\mathbf{x}^* = \left[\frac{\mathbf{x}_B^*}{\mathbf{0}}\right] = \left[\frac{\mathbf{B}^{-1}\mathbf{b}}{\mathbf{0}}\right] \quad \text{with an optimal cost } z^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$$

where $\mathbf{B}$ is the corresponding optimal basis matrix.

Since $\mathbf{x}_B^* = \mathbf{B}^{-1}\mathbf{b} > \mathbf{0}$, for a small enough incremental $\Delta\mathbf{b}$ in demand, we know $\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) > \mathbf{0}$ and

$$\bar{\mathbf{x}}^* = \left[\frac{\bar{\mathbf{x}}_B^*}{\mathbf{0}}\right] = \left[\frac{\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b})}{\mathbf{0}}\right]$$

is an optimal basic feasible solution (why?) to the following problem

$$\begin{aligned}&\text{Minimize} \quad \mathbf{c}^T\mathbf{x}\\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} + \Delta\mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}\end{aligned}$$

which is the same process of minimizing the total cost but satisfying more demands. Note that the optimal cost associated with this problem is $\bar{\mathbf{z}}^* = \mathbf{c}_B\mathbf{B}^{-1}(\mathbf{b}+\Delta\mathbf{b})$. Consequently,

$$\bar{\mathbf{z}}^* - z^* = \mathbf{c}_B\mathbf{B}^{-1}(\mathbf{b}+\Delta\mathbf{b}) - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_B\mathbf{B}^{-1}\Delta\mathbf{b} = (\mathbf{w}^*)^T\Delta\mathbf{b} \tag{4.17}$$

Recall that $\mathbf{w}^*$ is the simplex multiplier. At the primal optimal solution $\mathbf{x}^*$, it becomes the vector of dual variables. Equation (4.17) says the incremental cost $(\bar{\mathbf{z}}^* - \mathbf{z}^*)$ of satisfying an incremental demand $(\Delta\mathbf{b})$ is equal to $(\mathbf{w}^*)^T\Delta\mathbf{b}$. Therefore w_i^* can be thought as the *marginal cost* of the providing one unit of the ith demand at optimum. In other words, it indicates the minimum unit price one has to charge the customer for satisfying additional demands when an optimum is achieved. Therefore, the dual variables are sometimes called the *marginal prices*, the *shadow prices*, or the *equilibrium prices*.

4.4.2 Interpretation of the Dual Problem

This time, let us consider a linear programming problem in inequality form:

$$\begin{aligned}&\text{Maximize} \quad \mathbf{c}^T\mathbf{x}\\ &\text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}\end{aligned}$$

Its dual linear program becomes

$$\begin{aligned}&\text{Minimize} \quad \mathbf{b}^T\mathbf{w}\\ &\text{subject to} \quad \mathbf{A}^T\mathbf{w} \geq \mathbf{c}, \quad \mathbf{w} \geq \mathbf{0}\end{aligned}$$

First, let us explain the scenario of the primal linear program. Consider a manufacturer who makes n products out of m resources. To make one unit of product j $(j = 1, \ldots, n)$, it takes a_{ij} units of resource i for $i = 1, 2, \ldots, m$. The manufacturer has obtained b_i units of resource i $(i = 1, \ldots, m)$ in hand, and the unit price of product j $(j = 1, \ldots, n)$ is c_j at current market. Therefore, the primal problem leads the manufacturer to find an optimal production plan that maximizes the sales with available resources.

Next, we consider the dual scenario. Let us assume the manufacturer gets the resources from a supplier. The manufacturer wants to negotiate the unit purchasing price w_i for resource i $(i = 1, \ldots, m)$ with the supplier. Therefore the manufacturer's objective is to minimize the total purchasing price $\mathbf{b}^T\mathbf{w}$ in obtaining the resources b_i $(i = 1, \ldots, m)$. Since the marketing price c_j and the "product-resource" conversion ratio a_{ij} are open information on market, the manufacturer knows that, at least ideally, a "smart" supplier would like to charge him as much as possible, so that

$$a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m \geq c_j$$

In this way, the dual linear program leads the manufacturer to come up with a least-cost plan in which the purchasing prices are acceptable to the "smart" supplier.

The foregoing scenarios not only provide economic interpretations of the primal and dual linear programming problems, but also explain the implications of the complementary slackness conditions. Assume that the manufacturer already has b_i $(i = 1, \ldots, m)$ units of resources on hand. Then,

1. the ith component of the optimal dual vector w_i^* represents the maximum marginal price that the manufacturer is willing to pay in order to get an additional unit of resource i from the supplier;
2. when the ith resource is not fully utilized (i.e., $\mathbf{a}_i\mathbf{x}^* < b_i$ where $\mathbf{a}_i$ is the ith row of $\mathbf{A}$ and $\mathbf{x}^*$ is an optimal primal solution), the complementary slackness condition requires that $w_i^* = 0$, which means the manufacturer is not willing to pay a penny to get an additional amount of that resource;
3. when the supplier asks too much (i.e., when $A_j^T\mathbf{w}^* > c_j$, where $\mathbf{A}_j$ is the jth column of $\mathbf{A}$), the complementary slackness condition requires that $x_j^* = 0$, which means that the manufacturer is no longer willing to produce any amount of product j.

Many other interpretations of the dual variables, dual problems, and complementary slackness conditions can be found in the exercises.

4.5 THE DUAL SIMPLEX METHOD

With the concept of duality in mind, we now study a variant of the revised simplex method. Basically, this variant is equivalent to applying the revised simplex method to the dual linear program of a given linear programming problem. Hence we call it the *dual simplex method.*

4.5.1 Basic Idea of the Dual Simplex Method

Recall that the basic philosophy of the revised simplex method is to keep primal feasibility and complementary slackness conditions and seek for dual feasibility at its optimal solution. Similarly, the dual simplex method keeps dual feasibility and complementary slackness conditions but seeks for primal feasibility at its optimum.

Let us start with a basis matrix $\mathbf{B}$ which results in a dual feasible solution $\mathbf{w}$ such that

$$\mathbf{w}^T = {\mathbf{c}_B}^T\mathbf{B}^{-1} \quad \text{and} \quad \mathbf{A}^T\mathbf{w} \leq \mathbf{c} \tag{4.18}$$

We can further define

$$\mathbf{x} = \left[\frac{\mathbf{x}_B}{\mathbf{x}_N}\right] = \left[\frac{\mathbf{B}^{-1}\mathbf{b}}{\mathbf{0}}\right] \tag{4.19}$$

In this way, we see that

$$\mathbf{A}\mathbf{x} = [\mathbf{B} \mid \mathbf{N}] \left[\begin{array}{c} \mathbf{B}^{-1}\mathbf{b} \\ \hline \mathbf{0} \end{array}\right] = \mathbf{b} \tag{4.20}$$

and

$$\mathbf{r}^T\mathbf{x} = (\mathbf{c}^T - \mathbf{w}^T\mathbf{A})\mathbf{x} = \mathbf{c}^T\mathbf{x} - \mathbf{w}^T\mathbf{A}\mathbf{x} = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{0} \tag{4.21}$$

Therefore, the dual feasibility and complementary slackness conditions are satisfied in this setting. However, the primal feasibility is not satisfied unless $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$. In other words, before reaching an optimal solution, there exists at least one $p \in \tilde{\mathbf{B}}$ (the index set of basic variables in the primal problem) such that $x_p < 0$. The dual simplex method will reset $x_p = 0$ (that is, drop x_p from the basic variables) and choose an "appropriate" nonbasic variable $x_q \notin \tilde{\mathbf{B}}$ to enter the basis. Of course, during this pivoting process, the dual feasibility and and complementary slackness conditions should be maintained. This is the key idea behind the dual simplex method.

Note that the complementary slackness conditions are always satisfied because of the way we defined $\mathbf{w}$ and $\mathbf{x}$, hence we only have to concentrate on dual feasibility. Remember that, in Chapter 3, we showed that dual feasibility is associated with the reduced costs vector

$$\mathbf{r} = \left[\begin{array}{c} \mathbf{r}_B \\ \hline \mathbf{r}_N \end{array}\right]$$

with $\mathbf{r}_B = \mathbf{0}$ and $\mathbf{r}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$. Also remember that the fundamental matrix is

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} & \mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

with its inverse $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ Thus the information of dual variables and dual feasibility is embedded in the following equation:

$$(\mathbf{w}^T \mid \mathbf{r}_N{}^T) = \mathbf{c}^T\mathbf{M}^{-1} = (\mathbf{c}_B^T\mathbf{B}^{-1} \mid \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}) \tag{4.22}$$

Needless to say, after each pivoting a new basic variable is introduced to replace an old one, which results in a new fundamental matrix that produces new information on the dual according to Equation (4.22). Therefore, in order to maintain the dual feasibility, we exploit the matrix $\mathbf{M}^{-1}$ first.

4.5.2 Sherman-Morrison-Woodbury Formula

Note that the fundamental matrix $\mathbf{M}$ is an $n \times n$ matrix, and a direct inversion requires $O(n^3)$ elementary operations. In order to reduce the computational effort, also to reveal the new dual information in an explicit form, we introduce the Sherman-Morrison-Woodbury formula to modify the inverse of the fundamental matrix after each pivoting.

We first investigate the changes of the fundamental matrix (from $\mathbf{M}$ to $\overline{\mathbf{M}}$) after each pivoting. In this case, we assume that x_p leaves the basis and x_q enters the basis. Let $\mathbf{e}_j$ be an n-dimensional unit vector with 1 for its jth component and 0 for the rest. Then the new fundamental matrix $\overline{\mathbf{M}}$ can be obtained according to

$$\overline{\mathbf{M}} = \mathbf{M} + \mathbf{e}_q(\mathbf{e}_p - \mathbf{e}_q)^T \tag{4.23}$$

The following example illustrates this mechanism.

Example 4.4

Assume that $\mathbf{x}^T = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]$, x_1, x_2 are basic variables, x_3, x_4, x_5 are nonbasic, and, correspondingly,

$$\mathbf{A} = [\mathbf{B} \mid \mathbf{N}] = \left[\begin{array}{cc|ccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \end{array}\right]$$

and

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose that x_1 is leaving the basis ($p = 1$) and x_5 is entering the basis ($q = 5$). The new fundamental matrix is given by

$$\overline{\mathbf{M}} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [(1 \quad 0 \quad 0 \quad 0 \quad 0) - (0 \quad 0 \quad 0 \quad 0 \quad 1)]$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The inverse matrix of the new fundamental matrix can be obtained with the help of the following Sherman-Morrison-Woodbury formula.

Lemma 4.2. Let $\mathbf{M}$ be an $n \times n$ nonsingular matrix and $\mathbf{u}$, $\mathbf{v}$ be two n-dimensional column vectors. If $\omega = 1 + \mathbf{v}^T\mathbf{M}^{-1}\mathbf{u} \neq 0$, then the matrix $(\mathbf{M} + \mathbf{u}\mathbf{v}^T)$ is nonsingular and

$$(\mathbf{M} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{M}^{-1} - \left(\frac{1}{\omega}\right)\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{M}^{-1}$$

Proof.

$$\left[\mathbf{M}^{-1} - \left(\frac{1}{\omega}\right)\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{M}^{-1}\right]\left[\mathbf{M} + \mathbf{u}\mathbf{v}^T\right]$$

$$= \mathbf{I} - \left(\frac{1}{\omega}\right)\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T + \mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T - \left(\frac{1}{\omega}\right)\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T$$

$$= \mathbf{I} + \left(1 - \frac{1}{\omega}\right)\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T - \left(\frac{\omega - 1}{\omega}\right)\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T = \mathbf{I}$$

Note that once $\mathbf{M}^{-1}$ is known, the inverse matrix of $(\mathbf{M} + \mathbf{u}\mathbf{v}^T)$ can be found in $O(n^2)$ elementary operations. Sometimes, we call it the *rank-one* updating method.

To derive the inverse of the new fundamental matrix $\overline{\mathbf{M}}$, we let $\mathbf{u} = \mathbf{e}_q$, $\mathbf{v} = (\mathbf{e}_p - \mathbf{e}_q)$. Then, Lemma 4.2 implies that

$$\overline{\mathbf{M}}^{-1} = \mathbf{M}^{-1} - \frac{\mathbf{M}^{-1}\mathbf{e}_q(\mathbf{e}_p - \mathbf{e}_q)^T\mathbf{M}^{-1}}{1 + (\mathbf{e}_p - \mathbf{e}_q)^T\mathbf{M}^{-1}\mathbf{e}_q} \tag{4.24}$$

Notice that, $\mathbf{e}_q^T\mathbf{M}^{-1}$ is the qth row of $\mathbf{M}^{-1}$. Hence it is $(\mathbf{e}_q)^T$ itself. Consequently,

$$\overline{\mathbf{M}}^{-1} = \mathbf{M}^{-1} - \frac{\mathbf{M}^{-1}\mathbf{e}_q[\mathbf{e}_p^T\mathbf{M}^{-1} - \mathbf{e}_q^T]}{1 + \mathbf{e}_p^T\mathbf{M}^{-1}\mathbf{e}_q - \mathbf{e}_q^T\mathbf{e}_q}$$

$$= \mathbf{M}^{-1} - \frac{\mathbf{M}^{-1}\mathbf{e}_q[\mathbf{e}_p^T\mathbf{M}^{-1} - \mathbf{e}_q^T]}{\mathbf{e}_p^T\mathbf{M}^{-1}\mathbf{e}_q} \tag{4.25}$$

Remember that, from (4.22), $(\mathbf{w}^T \mid \mathbf{r}_N{}^T) = \mathbf{c}^T\mathbf{M}^{-1}$. We define

$$(\overline{\mathbf{w}}^T \mid \overline{\mathbf{r}}^T) = \mathbf{c}^T\overline{\mathbf{M}}^{-1} \tag{4.26}$$

Hence we have

$$\mathbf{c}^T\overline{\mathbf{M}}^{-1} = \mathbf{c}^T\mathbf{M}^{-1} - \frac{\mathbf{c}^T\mathbf{M}^{-1}\mathbf{e}_q[\mathbf{e}_p^T\mathbf{M}^{-1} - \mathbf{e}_q^T]}{\mathbf{e}_p^T\mathbf{M}^{-1}\mathbf{e}_q} \tag{4.27}$$

or

$$(\overline{\mathbf{w}}^T \mid \overline{\mathbf{r}}_N{}^T) = (\mathbf{w}^T \mid \mathbf{r}^T) - \frac{\mathbf{c}^T\mathbf{M}^{-1}\mathbf{e}_q[\mathbf{e}_p^T\mathbf{M}^{-1} - \mathbf{e}_q^T]}{\mathbf{e}_p^T\mathbf{M}^{-1}\mathbf{e}_q} \tag{4.28}$$

We further define

$$\mathbf{u}^T = \mathbf{e}_p^T\mathbf{B}^{-1} \tag{4.29}$$

$$y_j = \mathbf{u}^T\mathbf{A}_j \quad (\mathbf{A}_j \text{being the } j\text{th column of } \mathbf{A}) \tag{4.30}$$

and

$$\gamma = \frac{r_q}{y_q}. \tag{4.31}$$

Then Equation (4.28) shows that

$$\overline{\mathbf{w}} = \mathbf{w} + \gamma\mathbf{u} \tag{4.32}$$

$$\bar{r}_j = r_j - \gamma y_j, \qquad j \in \tilde{\mathbf{N}}, \quad j \neq q \quad \left(\begin{array}{l}\tilde{\mathbf{N}} \text{ being the index} \\ \text{set of nonbasic variables}\end{array}\right) \tag{4.33}$$

$$\bar{r}_p = -\gamma \tag{4.34}$$

Several observations can be made here:

1. Equation (4.29) says that $\mathbf{u}^T$ is the pth row of $\mathbf{B}^{-1}$.
2. Equation (3.30) further indicates that $y_q = \mathbf{u}^T\mathbf{A}_q = -d_p^q$, which is opposite to the pth component of the edge direction that we derived in the revised simplex method.
3. In order to maintain dual feasibility, we require

$$\bar{r}_p = -\gamma = -\frac{r_q}{y_q} \geq 0 \tag{4.35}$$

and

$$\bar{r}_j = r_j - \gamma y_j \geq 0 \qquad \text{for } j \in \tilde{\mathbf{N}} \tag{4.36}$$

If there exists $j \in \tilde{\mathbf{N}}$ such that $y_j < 0$, then $\frac{-r_j}{y_j} \geq -\gamma$ is required. Hence we must choose q such that

$$0 \leq -\gamma = \frac{-r_q}{y_q} \leq \frac{-r_j}{y_j}, \quad \forall\, y_j < 0, \quad j \in \tilde{\mathbf{N}} \tag{4.37}$$

In other words, we should choose q so that the *minimum ratio test*

$$0 \leq -\gamma = \frac{-r_q}{y_q} = \min\left\{\frac{-r_j}{y_j}\,\middle|\, y_j < 0, j \in \tilde{\mathbf{N}}\right\} \tag{4.38}$$

is satisfied.

4. In case $y_j \geq 0, \forall j \in \tilde{\mathbf{N}}$, then we know

$$y_j = \mathbf{u}^T\mathbf{A}_j = \mathbf{e}_p^T\mathbf{B}^{-1}\mathbf{A}_j \geq 0, \qquad \forall j \in \tilde{\mathbf{N}} \tag{4.39}$$

Therefore,

$$\mathbf{e}_p^T\mathbf{B}^{-1}\mathbf{A} = \mathbf{e}_p^T\mathbf{B}^{-1}[\mathbf{B} \mid \mathbf{N}] \geq 0. \tag{4.40}$$

Consequently, for any feasible $\mathbf{x} \geq \mathbf{0}$, we see that $\mathbf{e}_p^T\mathbf{B}^{-1}\mathbf{A}\mathbf{x} \geq 0$. Notice that $\mathbf{e}_p^T\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \mathbf{e}_p^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{e}_p^T\mathbf{x}_B = x_p$. Hence (4.39) implies that $x_p \geq 0$, which contradicts our assumption (of the dual simplex approach) that $x_p < 0$. This in turn implies that there is no feasible solution to the primal problem.

4.5.3 Computer Implementation of the Dual Simplex Method

Incorporating the above observations into the dual simplex approach, we can now present a step-by-step procedure of the dual simplex method for computer implementation. It solves a linear programming problem in its standard form.

Step 1 (starting with a dual feasible basic solution): Given a basis $\mathbf{B} = [\mathbf{A}_{j_1}, \mathbf{A}_{j_2}, \mathbf{A}_{j_3}, \dots \mathbf{A}_{j_m}]$ of the constraint matrix $\mathbf{A}$ in the primal problem with an index set $\tilde{\mathbf{B}} = \{j_1, j_2, j_3, \dots j_m\}$, such that a dual basic feasible solution $\mathbf{w}$ can be obtained by solving the system of linear equations

$$\mathbf{B}^T\mathbf{w} = \mathbf{c}_B$$

Compute the associated reduced costs vector $\mathbf{r}$ with

$$r_j = c_j - \mathbf{w}^T\mathbf{A}_j, \qquad \forall j \notin \tilde{\mathbf{B}}$$

Step 2 (checking for optimality): Compute vector $\mathbf{x}_B$ for primal basic variables by solving

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}$$

If $\mathbf{x}_B \geq \mathbf{0}$, then STOP. The current solution

$$\left[\frac{\mathbf{x}_B}{\mathbf{x}_N}\right] = \left[\frac{\mathbf{B}^{-1}\mathbf{b}}{\mathbf{0}}\right]$$

is optimal.

Otherwise go to Step 3.

Step 3 (leaving the basis): Choose a basic variable $x_{j_p} < 0$ with index $j_p \in \tilde{\mathbf{B}}$.

Step 4 (checking for infeasibility): Compute $\mathbf{u}$ by solving the system of linear equations

$$\mathbf{B}^T\mathbf{u} = \mathbf{e}_p$$

Also compute

$$y_j = \mathbf{u}^T\mathbf{A}_j, \qquad \forall j \notin \tilde{\mathbf{B}}$$

If $y_j \geq 0, \forall j \notin \tilde{\mathbf{B}}$; then STOP. The primal problem is infeasible. Otherwise go to Step 5.

Step 5 (entering the basis): Choose a nonbasic variable x_q by the minimum ratio test

$$\frac{-r_q}{y_q} = \min\left\{\frac{-r_j}{y_j} \;\middle|\; y_j < 0, j \notin \tilde{\mathbf{B}}\right\}$$

Set

$$\frac{-r_q}{y_q} = -\gamma$$

Step 6 (updating the reduced costs):

$$r_j \leftarrow r_j - \gamma y_j \qquad \forall j \notin \tilde{\mathbf{B}}, \quad j \neq q$$

$$r_{j_p} \leftarrow -\gamma$$

Step 7 (updating current solution and basis): Compute $\mathbf{d}$ by solving

$$\mathbf{B}\mathbf{d} = -\mathbf{A}_q$$

Set

$$x_q \leftarrow \alpha = \frac{x_{j_p}}{y_q} = \left(\frac{-x_{j_p}}{d_p}\right)$$

$$x_{j_i} \leftarrow x_{j_i} + \alpha d_{j_i}, \qquad \forall j_i \in \tilde{\mathbf{B}}, \quad i \neq p$$

$$\mathbf{B} \leftarrow \mathbf{B} + [A_q - A_{j_p}]\mathbf{e}_p^T$$

$$\tilde{\mathbf{B}} \leftarrow \tilde{\mathbf{B}} \cup \{q\} \backslash \{j_p\}$$

Go to Step 2.

The following example illustrates the dual simplex method.

Example 4.5

Consider the linear program

$$\begin{aligned}
\text{Minimize} \quad & -2x_1 - x_2 \\
\text{subject to} \quad & x_1 + x_2 + x_3 = 2 \\
& x_1 + x_4 \qquad = 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Step 1 (starting): Choose $\tilde{\mathbf{B}} = \{1, 4\}$. We see that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Then the dual solution

$$\mathbf{w} = \mathbf{c}_B^T\mathbf{B}^{-1} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Computing $r_j, \forall j \notin \tilde{\mathbf{B}}$, we have $r_2 = 1$ and $r_3 = 2$, which implies that $\mathbf{w}$ is dual feasible.

Step 2 (checking for optimality): Since

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

the corresponding primal vector is infeasible.

Step 3 (leaving the basis): Since $x_4 < 0$ (the second element in $\tilde{\mathbf{B}}$), we choose x_4 to be leaving the basis and let $p = 2$.

Step 4 (check infeasibility): Compute

$$\mathbf{u}^T = \mathbf{e}_2^T \mathbf{B}^{-1} = [-1 \quad 1]$$

and

$$y_2 = \mathbf{u}^T \mathbf{A}_2 = -1, \qquad y_3 = \mathbf{u}^T \mathbf{A}_3 = -1$$

Step 5 (entering the basis): Take the minimum ratio test

$$-\frac{r_2}{y_2} = \min\left\{\frac{-1}{-1}, \frac{-2}{-1}\right\} = 1 = -\gamma$$

Therefore x_2 is entering the basis and $p = 2$.

Step 6 (updating the reduced costs):

$$r_4 = -\gamma = 1 \quad \text{and} \quad r_3 = 2 - \gamma y_3 = 1$$

(also note that r_2 has been changed from 1 to 0 as x_2 enters the basis.)

Step 7 (updating current solution and basis): Solving for **d** in the equation $\mathbf{B}\mathbf{d} = -\mathbf{A}_2$, we obtain

$$\mathbf{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Also

$$x_2 = \alpha = \frac{x_4}{y_2} = 1$$

$$x_1 = 2 - 1 \times 1 = 1$$

Thus the new primal vector has $x_1 = x_2 = 1$ (and nonbasic variables $x_3 = x_4 = 0$). Since it is nonnegative, we know it is an optimal solution to the original linear program. The corresponding optimal basis **B** becomes

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

4.5.4 Find an Initial Dual Basic Feasible Solution

To start the dual simplex method, we need a basis matrix **B** which ensures a dual basic feasible solution. In contrast with the *artificial variable technique* introduced in Chapter 3 to obtain a starting primal basic feasible solution for the revised simplex algorithm, a popular method called the *artificial constraint technique* is used for the dual simplex method. Basically, we can choose any nonsingular $m \times m$ submatrix **B** of **A**, and add one artificial constraint

$$\sum_{i \in \bar{\mathbf{N}}} x_i \leq M$$

with a very large positive number M to the original problem. In this way, an additional slack variable x_{n+1} is added, and $\bar{\mathbf{B}} \cup \{n+1\}$ becomes an index set of basic variables for the new problem. Among those nonbasic variables, choose the one with minimum value in the reduced cost r_j as the entering variable and x_{n+1} as the leaving variable. It can be shown that a dual basic feasible solution can be identified by performing such a single pivot.

Another way to obtain a dual basic feasible solution is by solving the following linear programming problem (possible by applying the revised simplex method):

$$\text{Minimize} \quad \mathbf{c}^T \mathbf{x} \tag{4.41a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{e}, \ \mathbf{x} \geq \mathbf{0} \tag{4.41b}$$

where **B** is any $m \times m$ nonsingular submatrix of **A** and **e** is a vector of all ones. Note that problem (4.41) has a starting feasible solution

$$\begin{bmatrix} \mathbf{e} \\ \hline \mathbf{0} \end{bmatrix}$$

for the revised simplex method. If this leads to an optimal solution, the corresponding dual solution can be chosen as an initial dual basic feasible solution. On the other hand, if problem (4.41) becomes unbounded, we can show that the original linear program is also unbounded. Hence no dual feasible solution can be found. This is left as an exercise to the reader.

Before concluding this section, we would like to point out three facts:

1. Solving a linear program in its standard form by the dual simplex method is mathematically equivalent to solving its dual linear program by the revised (primal) simplex method.
2. Solving a linear program by the dual simplex method requires about the same amount of effort as the revised (primal) simplex method.
3. The dual simplex method is very handy in sensitivity analysis with an additional constraint. This topic will be discussed in later sections.

4.6 THE PRIMAL DUAL METHOD

As we discussed before, the dual simplex method starts with a basic feasible solution of the dual problem and defines a corresponding basic solution for the primal problem such that the complementary slackness conditions are met. Through a series of pivoting operations, the method maintains the dual feasibility and complementary slackness conditions and tries to attain the primal feasibility. Once the primal feasibility is achieved, the K-K-T optimality conditions guarantee an optimal solution. In this section, we study the so-called *primal-dual method*, which is very similar to the dual simplex approach but allows us to start with a nonbasic dual feasible solution.

Consider a linear programming problem in its standard form, which we may refer to as the "original problem." Let **w** be a dual feasible (possibly nonbasic) solution. Then we know that $c_j \geq \mathbf{w}^T \mathbf{A}_j \ \forall j$, where $\mathbf{A}_j$ represents the jth column of the constraint matrix **A**. We are particularly interested in those *binding (or tight)* constraints and denote an index set $\bar{\mathbf{T}} = \{j \mid \mathbf{w}^T \mathbf{A}_j = c_j\}$. According to the complementary slackness theorem

(Theorem 4.4), $\tilde{\mathbf{T}}$ is also the index set of primal variables which may assume positive values. Now we consider the following linear programming problem:

$$\text{Minimize } z = \sum_{j \in \tilde{\mathbf{T}}} 0x_j + \mathbf{e}^T \mathbf{x}^a \tag{4.42a}$$

$$\text{subject to } \sum_{j \in \tilde{\mathbf{T}}} \mathbf{A}_j x_j + \mathbf{I}\mathbf{x}^a = \mathbf{b} \tag{4.42b}$$

$$x_j \geq 0, \quad \forall j \in \tilde{\mathbf{T}}, \quad \text{and} \quad \mathbf{x}^a \geq \mathbf{0} \tag{4.42c}$$

where $\mathbf{x}^a$ is an m-dimensional vector of artificial variables.

Note that problem (4.42) only includes a subset of primal variables in the original problem, hence it is called the *restricted primal* problem associated with the original one. Also note that the following result is true.

Lemma 4.3. If the restricted primal problem has an optimal solution with zero objective value, then the solution must be an optimal solution to the original problem.

Proof. Assume that

$$\left[\begin{array}{c} \mathbf{x}_T^* \\ \hline \mathbf{x}_a^* \end{array}\right]$$

is an optimal solution to the restricted problem with zero objective value. Since the optimal objective value of the restricted primal problem is zero, we have $\mathbf{x}_a^* = \mathbf{0}$ in its optimal solution. Therefore we can use $\mathbf{x}_T^*$ to construct a primal feasible solution $\mathbf{x}$ to the original problem such that $x_j = x_j^* \geq 0, \forall j \in \tilde{\mathbf{T}}$, and $x_j = 0, \forall j \notin \tilde{\mathbf{T}}$. Note that the restricted problem was defined on the basis of an existing dual feasible solution $\mathbf{w}$ with $c_j = \mathbf{w}^T \mathbf{A}_j, \forall j \in \tilde{\mathbf{T}}$, and $c_j > \mathbf{w}^T \mathbf{A}_j, \forall j \notin \tilde{\mathbf{T}}$. It is clear that the complementary slackness conditions are satisfied in this case, since $(c_j - \mathbf{w}^T \mathbf{A}_j)x_j = 0, \forall j$. Thus the K-K-T conditions are satisfied and the proof is complete.

If the optimal objective value of the restricted primal problem is not zero, say $z^* > 0$, then $\mathbf{x}_T^*$ is not good enough to define a primal feasible solution to the original problem. In other words, a new dual feasible solution is needed to reconstruct the restricted primal problem with reduced value of z^*. In doing so, we also would like to make sure that only new primal variables whose index does not belong to $\tilde{\mathbf{T}}$ are passed on to the new restricted primal problem. To achieve our goal, let us consider the dual problem of the restricted primal problem (4.42), i.e.,

$$\text{Maximize } z' = \mathbf{y}^T \mathbf{b} \tag{4.43a}$$

$$\text{subject to } \mathbf{y}^T \mathbf{A}_j \leq 0, \qquad \forall j \in \tilde{\mathbf{T}} \tag{4.43b}$$

$$\mathbf{y} \leq \mathbf{e}, \qquad \mathbf{y} \quad \text{unrestricted} \tag{4.43c}$$

Let $\mathbf{y}^*$ be an optimal solution to this problem. Then the complementary slackness conditions imply that $\mathbf{y}^{*T} \mathbf{A}_j \leq 0$, for $j \in \tilde{\mathbf{T}}$. Only for those $j \notin \tilde{\mathbf{T}}$ with $\mathbf{y}^{*T} \mathbf{A}_j > 0$, the corresponding primal variable x_j could be passed on to the restricted primal problem with potential for lowering the value of z^*. (Why?) More precisely, we may consider $\mathbf{y}^*$ as a moving direction in translating the current dual feasible solution $\mathbf{w}$ to a new dual solution $\mathbf{w}'$, i.e., we define

$$\mathbf{w}' = \mathbf{w} + \alpha \mathbf{y}^*, \qquad \text{for} \quad \alpha > 0$$

Hence we have

$$c_j - \mathbf{w}'^T \mathbf{A}_j = c_j - (\mathbf{w} + \alpha \mathbf{y}^*)^T \mathbf{A}_j = (c_j - \mathbf{w}^T \mathbf{A}_j) - \alpha(\mathbf{y}^{*T} \mathbf{A}_j) \tag{4.44}$$

Now, for each $j \in \tilde{\mathbf{T}}$, since $c_j - \mathbf{w}^T \mathbf{A}_j = 0$ and $\mathbf{y}^{*T} \mathbf{A}_j \leq 0$, we know $c_j - \mathbf{w}'^T \mathbf{A}_j \geq 0$. In order to keep $\mathbf{w}'$ to be dual feasible, we have to consider those $j \notin \tilde{\mathbf{T}}$ with $\mathbf{y}^{*T} \mathbf{A}_j > 0$. Given the fact that $c_j - \mathbf{w}^T \mathbf{A}_j \geq 0, \forall j \notin \tilde{\mathbf{T}}$, we can properly choose $\alpha > 0$ according to the following formula:

$$\alpha = \frac{(c_k - \mathbf{w}^T \mathbf{A}_k)}{\mathbf{y}^{*T} \mathbf{A}_k} = \min_j \left\{ \frac{(c_j - \mathbf{w}^T \mathbf{A}_j)}{\mathbf{y}^{*T} \mathbf{A}_j} \;\middle|\; j \notin \tilde{\mathbf{T}}, \mathbf{y}^{*T} \mathbf{A}_j > 0 \right\} \tag{4.45}$$

such that $c_j - \mathbf{w}'^T A_j \geq 0, \forall j \notin \tilde{\mathbf{T}}$. In particular, $c_k - \mathbf{w}'^T \mathbf{A}_k = 0$ and $c_j - \mathbf{w}'^T \mathbf{A}_j \geq 0$, for $j \notin \tilde{\mathbf{T}}$ and $j \neq k$. Then the primal variable x_k is a candidate to enter the basis of the new restricted primal problem, in addition to those primal variables in the basis of the current restricted problem.

Following this process of adding primal variables into the restricted problem, we may end up with either one of the following two situations: Case 1—the optimal objective value of a new restricted primal problem becomes zero. Then Lemma 4.3 assures us an optimal solution to the original problem is reached. Case 2—the optimal objective value of a new restricted primal problem is still greater than zero but $\mathbf{y}^{*T} \mathbf{A}_j \leq 0, \forall j \notin \tilde{\mathbf{T}}$. Then we can show that the original primal problem is infeasible and its dual problem is unbounded.

4.6.1 Step-by-step Procedure for the Primal-Dual Simplex Method

Summarizing the discussions in the previous section, we can write down a step-by-step procedure for the primal-dual simplex method for computer implementation.

Step 1 (starting): Choose an initial dual vector $\mathbf{w}$ such that

$$c_j - \mathbf{w}^T \mathbf{A}_j \geq 0, \qquad \forall j$$

Let $\tilde{\mathbf{T}} = \{j \mid c_j - \mathbf{w}^T \mathbf{A}_j = 0\}$.

Step 2 (check for optimality): Solve the restricted primal problem (4.42). If the optimal cost of this problem is zero, then STOP. The current solution is optimal. Otherwise go to Step 3.

Step 3 (compute the direction of translation for the dual vector): Solve the dual problem (4.43) of the restricted primal problem. Let y^* be its optimal solution and take it as the direction of translation of the current dual solution.

Step 4 (check infeasibility/unboundedness): If $\mathbf{y}^{*T}\mathbf{A}_j \leq 0, \forall j \notin \tilde{\mathbf{T}}$, then STOP. The original primal problem is infeasible and its dual is unbounded. Otherwise, continue.

Step 5 (enter the basis of the restricted primal): Choose an index k such that

$$\frac{(c_k - \mathbf{w}^T\mathbf{A}_k)}{\mathbf{y}^{*T}\mathbf{A}_k} = \min_j \left\{ \frac{(c_j - \mathbf{w}^T\mathbf{A}_j)}{\mathbf{y}^{*T}\mathbf{A}_j} \,\middle|\, j \notin \tilde{\mathbf{T}}, \mathbf{y}^{*T}\mathbf{A}_j > 0 \right\}$$

Also define a step length

$$\alpha = \frac{(c_k - \mathbf{w}^T\mathbf{A}_k)}{\mathbf{y}^{*T}\mathbf{A}_k}$$

Add the primal variable x_k into the basis to form a new restricted primal problem.

Step 6 (update the dual feasible vector): Set

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha\mathbf{y}^*$$

Go to Step 1.

Note that the mechanisms of generating a starting dual feasible solution for the dual simplex method can be applied here to initiate the primal-dual method. The following example illustrates the procedures of the primal-dual algorithm.

Example 4.6

$$\begin{aligned} \text{Minimize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 2 \\ & x_1 + x_4 \qquad = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Step 1 (starting): The dual of the above problem is

$$\begin{aligned} \text{Maximize} \quad & 2w_1 + w_2 \\ \text{subject to} \quad & w_1 + w_2 \leq -2 \\ & w_1 \qquad \leq -1 \\ & w_1 \qquad \leq 0 \\ & w_2 \qquad \leq 0 \\ & w_1, w_2 \quad \text{unrestricted} \end{aligned}$$

Let us choose a dual feasible solution

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

Notice that only the second constraint is tight, hence $\tilde{\mathbf{T}} = \{2\}$.

Step 2 (check for optimality): The restricted primal is

$$\begin{aligned} \text{Minimize} \quad & x_1^a + x_2^a \\ \text{subject to} \quad & x_2 + x_1^a = 2 \\ & x_2^a = 1 \\ & x_2, x_1^a, x_2^a \geq 0 \end{aligned}$$

Solving it, we have an optimal solution $[x_2 \quad x_1^a \quad x_2^a]^T = [2 \quad 0 \quad 1]^T$ with the corresponding dual solution $[w_1 \quad w_2]^T = [0 \quad 1]^T$. Since the optimal cost is $1\ (\neq 0)$, the current solution is not optimal to the original problem.

Step 3 (compute the direction of translation for the dual vector): The dual to the restricted primal is

$$\begin{aligned} \text{Maximize} \quad & 2y_1 + y_2 \\ \text{subject to} \quad & y_1 \qquad \leq 0 \\ & y_1 \qquad \leq 1 \\ & y_2 \qquad \leq 1 \\ & y_1, y_2 \quad \text{unrestricted} \end{aligned}$$

Since x_2 and x_2^a are basic variables of the restricted primal, it follows the complementary slackness conditions that the first and third constraints of its dual problem are tight. Therefore,

$$\mathbf{y}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an optimal solution to this problem. We take it as the direction of translation for the dual vector $\mathbf{w}$.

Step 4 (check infeasibility/unboundedness): Now we proceed to compute the values $\mathbf{y}^{*T}\mathbf{A}_j$ for $j \in \{1, 3, 4\}$. It can be easily verified that these values are 1, 0, and 1 respectively. Therefore we continue.

Step 5 (enter the basis of the restricted primal): Compute $c_j - \mathbf{w}^T A_j$ for $j = 1, 3, 4$. The values are 2, 1, and 3 respectively. Therefore,

$$\alpha = \min\left\{\frac{2}{1}, \frac{3}{1}\right\} = 2$$

and $k = 1$. This implies that x_1 should also enter the basis in addition to x_2.

Step 6 (update the dual feasible vector): The new dual vector becomes

$$\begin{bmatrix} -1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

So far we have just completed one iteration and a new restricted primal problem is generated:

$$\begin{aligned}\text{Minimize } & x_1^a + x_2^a \\ \text{subject to } & x_1 + x_2 + x_1^a = 2 \\ & x_1 + x_2^a = 1 \\ & x_1, x_2, x_1^a, x_2^a \geq 0\end{aligned}$$

Solving it, we have an optimal solution $x_1 = x_2 = 1$ and $x_1^a = x_2^a = 0$ with a zero objective value. Hence we know $[1 \quad 1 \quad 0 \quad 0]^T$ is an optimal solution to the original problem and $[-1 \quad -1]^T$ is an optimal solution to its dual problem.

4.7 SENSITIVITY ANALYSIS

Given a linear programming problem in its standard form, the problem is completely specified by the constraint matrix **A**, the right-hand-side vector **b**, and the cost vector **c**. We assume that the linear programming problem has an optimal solution $\mathbf{x}^*$ for a given data set $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. In many cases, we find the data set $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ needs to be changed within a range after we obtained $\mathbf{x}^*$, and we are interested in finding out new optimal solutions accordingly. Conceptually, we can of course solve a set of linear programming problems, each one with a modified data value within the range. But this may become an extremely expensive task in reality. The knowledge of *sensitivity analysis* or *post-optimality analysis* will lead us to understand the implications of changing input data on the optimal solutions.

4.7.1 Change in the Cost Vector

Assume that $\mathbf{x}^*$ is an optimal solution with basis **B** and nonbasis **N** of a linear programming problem:

$$\begin{aligned}\text{Minimize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}\end{aligned}$$

Let $\mathbf{c}' = \left[\frac{\mathbf{c}'_B}{\mathbf{c}'_N}\right]$ be a *perturbation* in the cost vector such that the cost vector changes according to the formula

$$\bar{\mathbf{c}} = \mathbf{c} + \alpha\mathbf{c}' = \left[\frac{\mathbf{c_B}}{\mathbf{c_N}}\right] + \alpha\left[\frac{\mathbf{c}'_\mathbf{B}}{\mathbf{c}'_\mathbf{N}}\right] \tag{4.46}$$

where $\alpha \in R$.

We are specifically interested in finding out an upper bound $\overline{\alpha}$ and lower bound $\underline{\alpha}$ such that the current optimal solution $\mathbf{x}^*$ remains optimal for the linear programming problem with a new cost vector in which $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$. The geometric concept behind the effect of the above perturbation of **c** on $\mathbf{x}^*$ is illustrated in Figure 4.2. When the scale of perturbation is small, $\mathbf{x}^*$ may remain optimal. But a large-scale perturbation could lead to a different optimal solution.

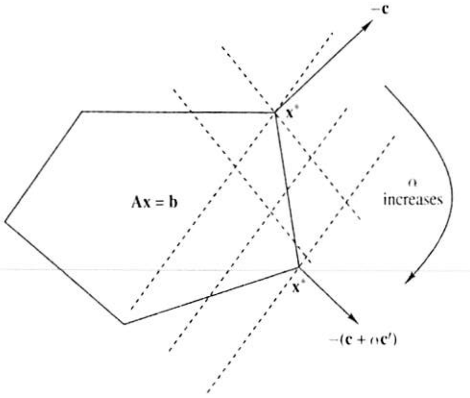

Figure 4.2

In order to find the *stable range* for the current optimal solution $\mathbf{x}^*$ with basis **B**, we focus for a moment on the revised simplex method. Notice that since the feasible domain $\{\mathbf{x} \in R^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ remains the same, $\mathbf{x}^*$ stays feasible in the linear program with the perturbed cost vector $\bar{\mathbf{c}}$. Moreover, $\mathbf{x}^*$ stays optimal if the reduced costs vector satisfies the requirement that

$$\bar{\mathbf{r}}_N^T = \bar{\mathbf{c}}_N^T - \bar{\mathbf{c}}_B^T\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0} \tag{4.47}$$

In other words, we require

$$(\mathbf{c}_N + \alpha\mathbf{c}'_N)^T - (\mathbf{c}_B + \alpha\mathbf{c}'_B)^T\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0} \tag{4.48}$$

We now define

$$\mathbf{r}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} \tag{4.49}$$

and

$$\mathbf{r}'^T_N = \mathbf{c}'^T_N - \mathbf{c}'^T_B\mathbf{B}^{-1}\mathbf{N} \tag{4.50}$$

Then, as long as α satisfies that

$$\alpha\mathbf{r}'^T_N \geq -\mathbf{r}_N^T, \tag{4.51}$$

$\mathbf{x}^*$ stays optimal for the linear programming problem with a perturbed cost vector $\bar{\mathbf{c}}$. Therefore, denoting $\tilde{\mathbf{N}}$ as the index set of nonbasic variables, we can determine that

$$\underline{\alpha} = \max\left\{\max\left\{\frac{-r_q}{r'_q} \,\middle|\, r'_q > 0, q \in \tilde{\mathbf{N}}\right\}, -\infty\right\} \tag{4.52}$$

and

$$\overline{\alpha} = \min\left\{ \min\left\{ \frac{-r_q}{r'_q} \,\middle|\, r'_q < 0, q \in \tilde{\mathbf{N}} \right\}, +\infty \right\} \tag{4.53}$$

Several observations can be made here:

Observation 1. For $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$, $\mathbf{x}^*$ remains as an optimal solution to the linear program with perturbed cost vector. Besides, the optimal objective value $z^*(\alpha)$ becomes a function of α such that

$$\begin{aligned} z^*(\alpha) &= (\mathbf{c}_B^T + \alpha \mathbf{c}'^T_B)\mathbf{B}^{-1}\mathbf{b} \\ &= z^* + \alpha(\mathbf{c}'^T_B \mathbf{B}^{-1}\mathbf{b}) \end{aligned} \tag{4.54}$$

which is a linear function of α, when α stays in the stable range.

Observation 2. If the perturbation is along any particular cost component, say c_j for $1 \leq j \leq n$, we can define $\mathbf{e}_j$ to be the vector with all zeros except one at its jth component and set $\mathbf{c}' = \mathbf{e}_j$. In this way, Equations (4.52) and (4.53) provide a stable range $[c_j + \underline{\alpha}, c_j + \overline{\alpha}]$ for the jth cost component. This also tells us how *sensitive* each cost coefficient is.

Observation 3. When α is within the stable range, the current solution $\mathbf{x}^*$ remains optimal and the optimal objective value is a linear piece in the range. As α goes beyond either the lower bound or the upper bound just a little bit, Figure 4.2 indicates that a neighboring vertex will become a new optimal solution with another stable range. This can be repeated again and again and the optimal objective function $z^*(\alpha)$ becomes a piecewise linear function. The piecewise linearity is between the bounds on α for various bases. We can further prove that $z^*(\alpha)$ is actually a concave piecewise linear function as shown in Figure 4.3.

4.7.2 Change in the Right-hand-side Vector

As in the previous section, let us assume that $\mathbf{x}^*$ is an optimal solution with basis $\mathbf{B}$ and nonbasis $\mathbf{N}$ to the linear programming problem

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

This time we incur a *perturbation* $\mathbf{b}'$ in the right-hand-side vector and consider the following linear program:

$$\text{Minimize} \quad z(\alpha) = \mathbf{c}^T\mathbf{x} \tag{4.55a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} + \alpha\mathbf{b}', \quad \mathbf{x} \geq \mathbf{0} \tag{4.55b}$$

for $\alpha \in R$.

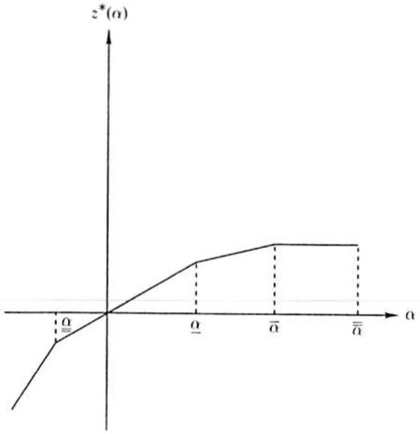

Figure 4.3

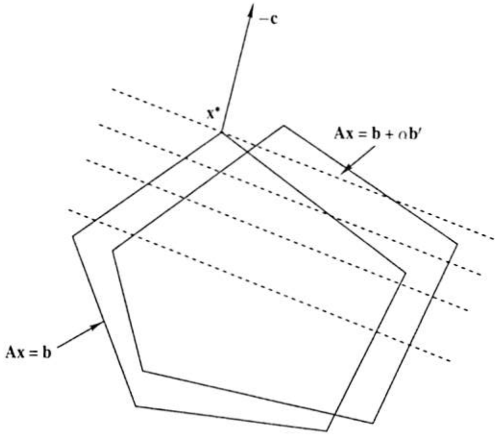

Figure 4.4

Note that because the right-hand-side vector has been changed, $\mathbf{x}^*$ need not be feasible any more. But we are specifically interested in finding an upper bound $\overline{\alpha}$ and lower bound $\underline{\alpha}$ such that the current basis $\mathbf{B}$ still serves as an optimal basis for the linear programming problem with a new right-hand-side vector in which $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$. The geometric implications of this problem are depicted in Figure 4.4.

In order to declare that $\mathbf{B}$ is an optimal basis, we have to check two conditions, namely,

1. The reduced cost vector $\mathbf{r}^T = \mathbf{c}_N{}^T - \mathbf{c}_B{}^T\mathbf{B}^{-1}\mathbf{N}$ is nonnegative.

2. The basic solution provided by **B** is feasible, i.e.,

$$\mathbf{x}_\alpha = \left[\frac{\mathbf{B}^{-1}(\mathbf{b}+\alpha\mathbf{b}')}{\mathbf{0}}\right] \geq \mathbf{0}$$

The first condition is obviously satisfied, since the cost vector **c**, the basis **B**, and the nonbasis **N** remain the same as before. The second condition is not necessarily true, owing to the change of $\alpha\mathbf{b}'$, unless $\mathbf{B}^{-1}(\mathbf{b}+\alpha\mathbf{b}') \geq \mathbf{0}$.

To find the *stable range* for α, we let $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$ and $\bar{\mathbf{b}}' = \mathbf{B}^{-1}\mathbf{b}'$. Thus $\bar{\mathbf{b}}+\alpha\bar{\mathbf{b}}' \geq \mathbf{0}$ is required for the second condition. Consequently, we can define

$$\underline{\alpha} = \max\left\{\max\left\{\frac{-\bar{b}_p}{\bar{b}'_p}\,\middle|\,\bar{b}'_p > 0,\ p \in \tilde{\mathbf{B}}\right\}, -\infty\right\} \tag{4.56}$$

and

$$\overline{\alpha} = \min\left\{\min\left\{\frac{-\bar{b}_p}{\bar{b}'_p}\,\middle|\,\bar{b}'_p < 0,\ p \in \tilde{\mathbf{B}}\right\}, +\infty\right\} \tag{4.57}$$

where $\tilde{\mathbf{B}}$ is the index set of the basic variables corresponding to **B**.

It can be clearly seen that within the range $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$, **B** remains an optimal basis for the perturbed linear program. Moreover, the corresponding optimal solutions

$$\mathbf{x}_\alpha = \left[\frac{\mathbf{B}^{-1}\mathbf{b}+\alpha\mathbf{B}^{-1}\mathbf{b}'}{\mathbf{0}}\right] = \left[\frac{\mathbf{B}^{-1}\mathbf{b}}{\mathbf{0}}\right] + \left[\frac{\alpha\mathbf{B}^{-1}\mathbf{b}'}{\mathbf{0}}\right] = \mathbf{x}^* + \alpha\mathbf{B}^{-1}\mathbf{b}'$$

form a linear function of α. In addition, the optimal objective values

$$\begin{aligned} z^*(\alpha) &= \mathbf{c}^T\mathbf{x}_\alpha = \mathbf{c}^T\mathbf{x}^* + \alpha\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}' \\ &= z^* + \alpha\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}' \end{aligned}$$

also become a linear function of α within the range.

If the perturbation is due to the change of the right-hand side of a particular constraint, say b_i for some $1 \leq i \leq m$, we can define $\mathbf{e}_i$ to be the vector with all zeros except one at its ith component and set $\mathbf{b}' = \mathbf{e}_i$. In this way, Equations (4.56) and (4.57) provide a stable range $[b_i + \underline{\alpha}, b_i + \overline{\alpha}]$ for the ith resource constraint, which indicates how *sensitive* the resource is.

4.7.3 Change in the Constraint Matrix

So far, we have dealt with the changes in the cost vector and the right-hand-side vector. In this section, we proceed to analyze the situation with changes in the constraint matrix. In general, the changes made in the constraint matrix may result in different optimal basis and optimal solutions. It is not a simple task to perform the sensitivity analysis. Here we deal only with four simpler cases, namely adding and removing a variable and adding and removing a constraint. As in previous sections, we still assume that the original linear programming problem has an optimal solution $\mathbf{x}^* = [\mathbf{B}^{-1}\mathbf{b}\,|\,\mathbf{0}]$ with an optimal basis **B** such that the constraint matrix can be partitioned as $\mathbf{A} = [\mathbf{B}\,|\,\mathbf{N}]$.

Case 1 (adding a new variable). Suppose that a new decision variable, say x_{n+1}, is identified after we obtained the optimal solution $\mathbf{x}^*$ of the original linear program. Let us also assume that c_{n+1} is the cost coefficient associated with x_{n+1}, and $\mathbf{A}_{n+1}$ is the associated column in the new constraint matrix. We would like to find an optimal solution to the new linear programming problem:

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} + c_{n+1}x_{n+1} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} + \mathbf{A}_{n+1}x_{n+1} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \quad x_{n+1} \geq 0 \end{aligned}$$

Note that we can set $x_{n+1} = 0$; then

$$\left[\frac{\mathbf{x}^*}{0}\right]$$

becomes a basic feasible solution to the new linear program. Hence the simplex algorithm can be initiated right away. Remember that $\mathbf{x}^*$ is an optimal solution to the original problem, the reduced costs r_j, for $j = 1, \ldots,$ must remain nonnegative. Therefore, we only have to check the additional reduced cost $r_{n+1} = c_{n+1} - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_{n+1}$.

If $r_{n+1} \geq 0$, then the current solution $\mathbf{x}^*$ with $x_{n+1} = 0$ is an optimal solution to the new problem and we do not have to do anything. On the other hand, if $r_{n+1} < 0$, then x_{n+1} should be included in the basis as a basic variable. Therefore, we can continue the simplex algorithm to find an optimal solution to the new linear programming problem.

Case 2 (removing a variable). After solving a linear programming problem, we find that a decision variable, say x_k, is no longer available and hence has to be removed from consideration. Our objective is to find a new optimal solution with minimum additional effort.

Note that if $x_k^* = 0$, then the current optimal solution $\mathbf{x}^*$ remains optimal. When $x_k^* > 0$ (x_k is a basic variable), we have to work out a new solution. In this case, we first attempt to remove x_k from the basis by solving the following Phase I problem:

$$\begin{aligned} &\text{Minimize} \quad x_k \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Since the constraints are not altered, we know $\mathbf{x}^*$ can be served as an initial basic feasible solution to this problem for the revised simplex algorithm. Moreover, if the simplex method finds the optimal objective value of the Phase I problem is not zero, then the new linear programming problem obtained by removing the variable x_k from the original problem must be infeasible. On the other hand, if the simplex method finds an optimal solution $\mathbf{x}'$ with zero objective value for the Phase I problem, then we can take $\mathbf{x}'$ as an initial basic feasible solution to the new linear program without the variable x_k. In a finite number of iterations, either an optimal solution can be found for this new problem, or the unboundedness can be detected.

Case 3 (adding a constraint). This time a new constraint is imposed after solving a linear programming problem. For simplicity, we assume the additional constraint has inequality form, namely,

$$\mathbf{a}_{m+1}^T\mathbf{x} \le b_{m+1} \tag{4.58}$$

where $\mathbf{a}_{m+1}{}^T$ is an n-dimensional row vector to be added to the constraint matrix **A**.

Hence the new linear problem becomes

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \\ &\qquad \mathbf{a}_{m+1}^T\mathbf{x} \le b_{m+1}, \quad \mathbf{x} \ge \mathbf{0} \end{aligned}$$

To solve this new linear program, first notice that the additional constraint may cut the original feasible domain to be smaller. If $\mathbf{x}^*$ remains feasible, then of course it remains optimal. But the feasible domain may exclude $\mathbf{x}^*$, as shown in Figure 4.5. In this case, we do not even have a basic feasible solution to start the simplex algorithm. Also notice that **B** is no longer a basis in the new problem. In fact, if the additional constraint is not redundant, the dimensionality of any new basis becomes $m+1$, instead of m.

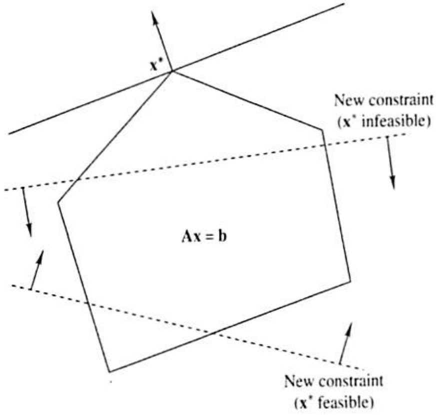

Figure 4.5

To solve the new problem with an additional constraint, we add a slack variable x_{n+1} and consider the following linear programming problem:

$$\text{Minimize} \quad \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N + 0x_{n+1} \tag{4.59a}$$

$$\text{subject to} \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \tag{4.59b}$$

$$(\mathbf{a}_{m+1_B})^T\mathbf{x}_B + (\mathbf{a}_{m+1_N})^T\mathbf{x}_N + x_{n+1} = b_{m+1} \tag{4.59c}$$

$$\mathbf{x}_B, \mathbf{x}_N \ge \mathbf{0}, x_{n+1} \ge 0 \tag{4.59d}$$

---

where $\mathbf{a}_{m+1_B}$ and $\mathbf{a}_{m+1_N}$ are the subrows of $\mathbf{a}_{m+1}$ corresponding to $\mathbf{x}_B$ and $\mathbf{x}_N$, respectively.

We now pass the slack variable to the basis **B** and consider a new basis $\overline{\mathbf{B}}$ defined by

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{a}_{m+1_B}^T & 1 \end{bmatrix} \tag{4.60}$$

It is easy to verify that $\overline{\mathbf{B}}$ is nonsingular and its inverse matrix is given by

$$\overline{\mathbf{B}}^{-1} = \begin{bmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -(\mathbf{a}_{m+1_B}^T\mathbf{B}^{-1}) & 1 \end{bmatrix} \tag{4.61}$$

With the new basis $\overline{\mathbf{B}}$, we can define

$$\overline{\mathbf{x}}_B = \overline{\mathbf{B}}^{-1}\left[\frac{\mathbf{b}}{\mathbf{b}_{m+1}}\right] \tag{4.62}$$

Then

$$\overline{\mathbf{x}} = \left[\frac{\overline{\mathbf{x}}_B}{0}\right]$$

is a basic solution (not necessarily feasible) to the new problem with an additional constraint. Moreover, we can show the following result.

Lemma 4.4. Let **B** be an optimal basis to the original linear programming problem. If $\overline{\mathbf{x}}$, essentially defined by (4.62), is nonnegative, then it is an optimal solution to the new linear programming problem with the additional constraint.

Proof. Since the basic solution $\overline{\mathbf{x}}$ is nonnegative, it is a basic feasible solution. In order to declare it is an optimal solution, we need to show the reduced cost for each nonbasic variable is nonnegative, i.e.,

$$c_q - \left[\frac{\mathbf{c}_B}{\mathbf{0}}\right]^T \overline{\mathbf{B}}^{-1}\left[\frac{\mathbf{A}_q}{a_{m+1,q}}\right] \ge 0, \qquad \forall\, q \in \tilde{\mathbf{N}} \tag{4.63}$$

Since **B** is an optimal basis to the original linear program, we have

$$c_q - \mathbf{c}_B{}^T\mathbf{B}^{-1}\mathbf{A}_q \ge 0 \qquad \forall\, q \in \tilde{\mathbf{N}} \tag{4.64}$$

Noting that

$$\left[\frac{\mathbf{c}_B}{0}\right]^T \overline{\mathbf{B}}^{-1} = [\mathbf{c}_B^T\mathbf{B}^{-1} \,|\, \mathbf{0}]$$

we see that condition (4.63) is true and the proof is completed.

On the other hand, if $\overline{\mathbf{x}}_B$ is not nonnegative, then the primal feasibility condition is violated by at least one primal variable. In this case, we can restore the primal feasibility condition by employing the dual simplex algorithm, starting with a dual basic feasible solution

$$\left[\frac{\mathbf{w}}{0}\right]$$

where $\mathbf{w}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$. The following example illustrates this situation.

Example 4.7

Consider the problem,

$$\begin{aligned} \text{Minimize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

It is easy to verify that $x_1^* = 2$, $x_2^* = x_3^* = 0$ is the optimal solution to this problem with an optimal basis $\mathbf{B} = [1]$. Moreover, we have the index set of basic variables $\tilde{\mathbf{B}} = \{1\}$, the nonbasic index set $\tilde{\mathbf{N}} = \{2, 3\}$, and $\mathbf{c}_B = [-2]$.

One more constraint is added to form a new linear program

$$\begin{aligned} \text{Minimize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 2 \\ & x_1 + + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

It is clear that $\mathbf{x}^*$ becomes infeasible to the new problem.

We now form a new basis

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

with $\mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. The dual solution is defined by $\mathbf{w} = \mathbf{c}_B^T \mathbf{B}^{-1}$. For the reduced costs r_j $(j = 2, 3)$, we have $r_2 = 1$ and $r_3 = 2$, which implies that $\mathbf{w}$ is dual feasible. However,

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

we know the corresponding primal is infeasible. Therefore we can restore the primal feasibility by the dual simplex method. The rest follows Example 4.5 exactly to the end.

Case 4 (removing a constraint). This case is more complicated than the ones we have considered so far. However, if the constraint, say $\mathbf{a}_k{}^T \mathbf{x} \leq b_k$, that we wish to remove is nonbinding, i.e., $\mathbf{a}_k^T \mathbf{x} < b_k$, then it can be removed without affecting the optimality of the current optimal solution. To check if the kth constraint is binding, we simply look at the dual variable w_k. If $w_k = 0$, then the complementary slackness condition allows the constraint to be not binding.

On the other hand, if we want to remove a binding constraint, the task becomes difficult. We may have to solve the new linear programming problem from the beginning.

4.8 CONCLUDING REMARKS

In this chapter, we have introduced the fundamental concept of duality theory in linear programming. Two variants of the simplex algorithm, namely the dual simplex algorithm and the primal-dual algorithm, have been derived based on this very concept. We also studied post-optimality analysis, which could assess the sensitivity of an optimal solution or optimal basis with respect to various changes made in the input data of a linear programming problem.

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EXERCISES

4.1. Prove that the symmetric pair in Example 4.2 are indeed a pair of primal and dual problems by converting the primal problem into its standard form first.

4.2. Find the linear dual program of the following problems:

(a) Minimize $9x_1 + 6x_2$

$$\begin{aligned} \text{subject to} \quad & 3x_1 + 8x_2 \geq 4 \\ & 5x_1 + 2x_2 \geq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(b) Maximize $4x_1 + 7x_2$

subject to $3x_1 + 5x_2 \leq 9$

$8x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Combining the results of (a) and (b), what's your conclusion?

4.3. Show that (a-1) and (a-2), (b-1) and (b-2) are primal-dual pairs:

(a-1) Minimize $\mathbf{c}^T\mathbf{x}$

subject to $\mathbf{Ax} = \mathbf{b}$

(a-2) Maximize $\mathbf{b}^T\mathbf{w}$

subject to $\mathbf{A}^T\mathbf{w} = \mathbf{c}$

(b-1) Minimize $\mathbf{c}^T\mathbf{x}$

subject to $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \leq \mathbf{0}$

(b-2) Maximize $\mathbf{b}^T\mathbf{w}$

subject to $\mathbf{A}^T\mathbf{w} \geq \mathbf{c}, \mathbf{w} \leq \mathbf{0}$

4.4. Find the dual linear program of the following problem:

$$\begin{aligned} \text{Minimize} \quad & 9x_1 + 6x_2 - 4x_3 + 100 \\ \text{subject to} \quad & 3x_1 + 8x_2 - 5x_3 \geq 14 \\ & 5x_1 - 2x_2 + 6x_3 = 17 \\ & 2x_1 + 4x_2 \leq 19 \\ & x_1 \leq 0, x_2 \geq 0, x_3 \quad \textit{unrestricted} \end{aligned}$$

4.5. Find the dual problems of the following linear programming problems:

(a) Minimize $\mathbf{c}^T\mathbf{x}$

subject to $\mathbf{Ax} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$

(b) Maximize $\mathbf{b}^T\mathbf{w}$

subject to $\mathbf{A}^T\mathbf{w} \leq \mathbf{c}, \quad \mathbf{w} \geq \mathbf{0}$

(c) Minimize $\mathbf{c}^T\mathbf{x}$

subject to $\mathbf{Ax} = \mathbf{b}, \quad \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$

(**l** and **u** are vectors of lower bounds and upper bounds.)

(d) Minimize $\mathbf{c}^T\mathbf{x}$

subject to $\mathbf{Ax} = \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1, \quad \mathbf{x} \geq \mathbf{0}$

(This is the famous Karmarkar's standard form which will be studied in Chapter 6.)

(e) Minimize $\sum_{i=1}^{n} x_i$

subject to $x_i - \sum_{j=1}^{n} \alpha \mathbf{P}_{ij}^k x_j \geq R_i^k, \quad \forall i = 1, 2, \cdots, N, \quad \forall k = 1, 2, \cdots, l$

x_i unrestricted

where $\mathbf{P}^k$ is an $N \times N$ (probability) matrix for $k = 1, 2, \ldots, l \alpha \in (0, 1), R_i^k \in R^+, \forall i, k$. (This is the *policy iteration* problem in dynamic programming.)

4.6. Construct an example to show that both the primal and dual linear problems have no feasible solutions. This indicates that the infeasibility of one problem does not imply the unboundedness of the other one in a primal-dual pair.

4.7. For an infeasible linear program, show that if its dual linear program is feasible, then the dual must be unbounded.

4.8. For a linear programming problem (P) in its standard form, assume that $\mathbf{A}$ is an $m \times n$ matrix with full row rank. Answer the following questions with reasons.

(a) For each basis $\mathbf{B}$, let $\mathbf{w}^T(\mathbf{B}) = \mathbf{c}_B^T\mathbf{B}^{-1}$ be the vector of simplex multipliers. Is $\mathbf{w}(\mathbf{B})$ always a feasible solution to its dual problem?

(b) Can every dual feasible solution be represented as $\mathbf{w}^T(\mathbf{B}) = \mathbf{c}_B^T\mathbf{B}^{-1}$ for some basis $\mathbf{B}$?

(c) Since $\mathbf{A}$ has full row rank, can we guarantee that (P) is nondegenerate?

(d) If (P) is nondegenerate, can we guarantee that its dual is also nondegenerate?

(e) Is it possible that both (P) and its dual are degenerate?

(f) Is it possible that (P) has a unique optimal solution with finite objective value but its dual problem is infeasible?

(g) Is it possible that both (P) and its dual are unbounded?

(h) When (P) and its dual are both feasible, show that the duality gap vanishes.

4.9. Consider a *two-person zero-sum game* with the following pay-off matrix to the row player:

Strategies	1	2	3
1	2	−1	0
2	−3	1	1

(This means the row player has two strategies and the column player has three strategies. If the row player chooses his/her second strategy and the column player chooses his/her third strategy, then the column player has to pay the row player $1.)

Let x_1, x_2, and x_3 be the probabilities with which the column player selects his/her first, second, and third strategies over many plays of the game. Keep in mind that the column player wishes to minimize the maximal expected payoff to the row player.

(a) What linear program will help the column player to determine his probability distribution of selecting different strategies?

(b) Find the dual problem of the above linear program.

(c) Interpret the dual linear program.
(d) Solve the dual linear program graphically.
(e) Use the dual optimal solution to compute the column player's probabilities.
(f) Write down and interpret the complementary slackness conditions for the two-person zero-sum game.

4.10. Here is a description of the transportation problem:

A company needs to ship a product from m locations to n destinations. Suppose that a_i units of the product are available at the ith origin $(i = 1, 2, 3, \ldots, n)$, b_j units are required at the jth destination $(j = 1, 2, \ldots n)$. Assume that the total amount of available units at all origins equals the total amount required at all destinations. The cost of shipping one unit of product from origin i to destination j is c_{ij} and you are asked to minimize the transportation cost.

(a) Formulate the problem as a linear programming problem.
(b) Write its dual linear program.
(c) Write down its complementary slackness conditions.
(d) Given that $i = 3$, $j = 4$, $a_1 = 3$, $a_2 = 3$, $a_4 = 4$; $b_1 = 2$, $b_2 = 3$, $b_3 = 2$, and $b_4 = 3$ with the cost matrix

		Destination			
		1	2	3	4
Origin	1	7	2	−2	8
	2	19	5	−2	12
	3	5	8	−9	3

and assuming that $\mathbf{w} = (0, 3, -4, 7, 2, -5, 7)^T$ is an optimal dual solution, find an optimal solution to the original (primal) problem.

4.11. Closely related to Farka's theorem of alternatives is Farka's transposition theorem: "There is a solution $\mathbf{x}$ to the linear system $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ if, and only if, $\mathbf{b}^T\mathbf{w} \geq \mathbf{0}$ when $\mathbf{A}^T\mathbf{w} \geq \mathbf{0}$." Prove Farka's transposition theorem.

4.12. Show that there is a solution $\mathbf{x}$ to the linear system $\mathbf{Ax} \leq \mathbf{b}$ if, and only if, $\mathbf{b}^T\mathbf{w} \geq \mathbf{0}$ when $\mathbf{A}^T\mathbf{w} = \mathbf{0}$ and $\mathbf{w} \geq \mathbf{0}$. This problem is called Gale's transposition theorem.

4.13. Show that there is a solution $\mathbf{x}$ to the linear system $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ if, and only if, $\mathbf{b}^T\mathbf{w} \geq \mathbf{0}$ when $\mathbf{A}^T\mathbf{w} \geq \mathbf{0}$ and $\mathbf{w} \geq \mathbf{0}$.

4.14. Prove Gordan's transposition theorem: There is a solution $\mathbf{x}$ to the strict homogeneous linear system $\mathbf{Ax} < \mathbf{0}$ if, and only if, $\mathbf{w} = \mathbf{0}$ when $\mathbf{A}^T\mathbf{w} = \mathbf{0}$ and $\mathbf{w} \geq \mathbf{0}$.

4.15. Use Farka's lemma to construct a proof of the strong duality theorem of linear programming.

4.16. Why is $\overline{\mathbf{x}}^*$ an optimal solution to the linear programming problem with new demands in Section 4.4.1 ?

4.17. Show that, in applying the primal-dual method, if we end with a restricted primal problem with positive optimal objective value and $\mathbf{y}^{*T}\mathbf{A}_j \leq 0, \forall j \notin \tilde{\mathbf{T}}$, then the original primal problem is infeasible and its dual is unbounded.

4.18. Consider the following linear program:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

First, use the revised simplex method to find the optimal solution and its optimal dual variables. Then use sensitivity analysis to answer the following questions.

(a) Find a new optimal solution if the cost coefficient of x_2 is changed from 1 to 6.
(b) Find a new optimal solution if the coefficient of x_2 in the first constraint is changed from 2 to 0.25
(c) Find a new optimal solution if we add one more constraint $x_2 + x_3 = 3$.
(d) If you were to choose between increasing the right-hand side of the first and the second constraints, which one would you choose? Why? What is the effect of this increase on the optimal value of the objective function?
(e) Suppose that a new activity x_6 is proposed with a unit cost of 4 and a consumption vector $\mathbf{A}_6 = (1 \quad 2)^T$. Find a corresponding optimal solution.