Issued: 09/24/2024 OR/MA/ST 706 Homework #3 Due: 10/01/2024 in class

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1. (i) (5 points) Is the function $f(x) = e^x$ a convex function defined on R? Why? Draw the graph of $f(x)$ over R.

   (ii) (5 points) Given $c_0, c_1, ..., c_n \in R$, when will the function $g(\mathbf{x}) = c_0 e^{\sum_{j=1}^{n} c_j x_j}$ be a convex function over E^n ? Why?

   (iii) (10 points) Define that $0 \log 0 = 0$. Let $c > 0$ and $h(y) = y \log \frac{y}{c}$ be a function defined on $\Omega = \{y \in R \mid y \geq 0\}$. Is $g(y)$ a convex function on Ω? Why? Draw the graph of $h(y)$ with $c = 1$ over Ω. Find the global minimum solution of $h(y)$ with $c = 1$ over Ω.

   (iv) (10 points) Let $f_1(x) = x - 1,\ f_2(x) = 2x - 2,\ f_3(x) = -x - 1,\ f_4(x) = -2x - 2$, and $p(x) = max\{f_1(x), f_2(x), f_3(x), f_4(x)\}$ for $x \in R$. Is $p(x)$ a convex function over R? Why? Draw the graph of $p(x)$ over R. Find the global minimum solution of $h(x)$ over R, if there exists one.

2. (i) (10 points) Let $f(\mathbf{x}) = ||\mathbf{x}||_2 = \sqrt{x_1^2 + ... + x_n^2}$ be the 2-norm function defined on E^n. Is the function $f(\mathbf{x})$ a convex function on E^n? Why? What's the gradient function of $f(\mathbf{x})$ over E^n? What't the Hessian function of $f(\mathbf{x})$ over E^n. Find the global minimum solution of $f(\mathbf{x})$ over E^n, if there exists one.

   (ii) (10 points) Let $g(\mathbf{x}) = ||\mathbf{x}||_2^2$ be the square of 2-norm function defined on E^n. Is the function $g(\mathbf{x})$ a convex function on E^n? Why? What's the gradient function of $g(\mathbf{x})$ over E^n? What't the Hessian function of $g(\mathbf{x})$ over E^n. Find the global minimum solution of $g(\mathbf{x})$ over E^n, if there exists one.

   (iii) (10 points) Let $h(\mathbf{x}) = ||A\mathbf{x}||_2^2$, with A being an $m \times n$ matrix, defined on E^n. Is the function $h(\mathbf{x})$ a convex function on E^n? Why? What's the gradient function of $h(\mathbf{x})$ over E^n? What't the Hessian function of $h(\mathbf{x})$ over E^n. Find the global minimum solution of $h(\mathbf{x})$ over E^n, if there exists one.

3. (i) (30 points) It's the time for you to sharpen your skill of proving things mathematically. Please try to prove the following claims by yourself first. If you have difficulties of doing so, you can find formal proofs in any reference book that may lead you to learn more about proving things by yourself. More proofs are waiting for you down the road in this class!

   "Let $\Omega \subset E^n$ be a convex set, $f_1, f_2 : \Omega \to R$ be convex functions. Then

   (a) $f_1 + f_2$ is convex on Ω.

   (b) βf_1 is convex on Ω, $\forall\ \beta \geq 0$."

   (c) $max\{f_1, f_2\}$ is convex on Ω.

   (ii) (10 points) Other than convex functions, is there any function whose local minimum over a convex set $S \in E^n$ is always a global minimum over S?

   If your answer is "Yes", show me a couple of such functions.

   If your answer is "No", please tell me your reasons.