$OR/MA/ST$ 706 *Homework* #4

Issue date: 10/23/2024 **Due date: 10/31/2024**

1. (20 points)
   Use the KKT conditions to find the local optimum solutions of the following problem:
   $$\begin{aligned} &Minimize \quad f(x) = x_1^2 - x_2 + 1 \\ &\text{s.t.} \quad h(x) = x_1^2 + x_2^2 = 1 \\ &\quad x \in \Omega = E^2 \end{aligned}$$
2. (20 points)
   Use the KKT conditions to find the local optimum solutions of the following problem:
   $$\begin{aligned} &Minimize \quad f(x) = x_1^2 - x_2 + 1 \\ &\text{s.t.} \quad g(x) = x_1^2 + x_2^2 \leq 1 \\ &\quad x \in \Omega = E^2 \end{aligned}$$
3. (20 points)
   Use the KKT conditions to find the local optimum solutions of the following problem:
   $$\begin{aligned} &Minimize \quad f(x) = x_1^2 - x_2 + 1 \\ &\text{s.t.} \quad h(x) = x_2 - \frac{\sqrt{2}}{2} = 0 \\ &\quad g(x) = x_1^2 + x_2^2 \leq 1 \\ &\quad x \in \Omega = E^2 \end{aligned}$$
4. (10 points)
   Let $h(\cdot): E^n \to E^1$ be a real-valued function and $d \in E^n$ be a vector of moving direction. We know that $h(x) = 0$ represents an $(n-1) -$ *dimensional* surface S in E^n.

   Is the following statement true or false? Why?
   "At a point $\bar{x} \in E^n$ lying on the surface S, i.e., $h(\bar{x}) = 0$, d is a feasible direction for $\bar{x}$ to move **if and only if** $\nabla h(\bar{x})\, d = 0$."
5. (10 points)
   Let $g(\cdot): E^n \to E^1$ be a real-valued function and $d \in E^n$ be a vector of moving direction. We know that $g(x) \leq 0$ represents an $n -$ *dimensional* "half space" in E^n.

   Is the following statement true or false? Why?
   "Assume that $g(\cdot)$ is an active constraint at a point $\bar{x} \in E^n$, i.e., $g(\bar{x}) = 0$, d is a feasible direction for $\bar{x}$ to move if and only if $\nabla g(\bar{x})\, d \leq 0$."

6. (20 points)

(a) For the KKT Theorem stated in Lecture 6 (Slide #11), interpret the meaning of the first requirement in English, i.e.,

" $\nabla f(x^*) + \lambda^T \nabla h(x^*) + \mu^T \nabla g(x^*) = 0$ with $\lambda \in E^m$ *and* $\mu \in E_+^p$ ".

Remember that $h(x) = \left(h_1(x), \dots, h_m(x)\right)^T$ and $g(x) = \left(g_1(x), \dots, g_p(x)\right)^T$.

(b) For the KKT Theorem stated in Lecture 6 (Slide #11), interpret the meaning of the second requirement in English, i.e.,

" $\mu^T g(x^*) = 0$ with $g(x^*) \le 0$ *and* $\mu \in E_+^p$ ".

(c) Putting (a) and (b) together, what is the role of the Lagrange multiplies λ *and* μ playing in the KKT conditions?