$OR/MA/ST$ 706 $Homework$ #5

Issue date: 10/31/2024 **Due date: 11/07/2024**

1. (15 points)
   Find the Lagrangian dual problem of the following problem:

$$\begin{aligned} Minimize \quad & f(x) = x_1^2 - x_2 \\ \text{s.t.} \quad & g(x) = x_1^2 + x_2^2 \leq 1 \\ & x \in X = E^2 \end{aligned}$$

   Is there any duality gap between the primal problem and dual problem? Why?

2. (15 points)
   Find the Lagrangian dual problem of the following problem:

$$\begin{aligned} Minimize \quad & f(x) = x_1^2 - x_2 \\ \text{s.t.} \quad & h(x) = x_1^2 + x_2^2 = 1 \\ & x \in X = E^2 \end{aligned}$$

   Is there any duality gap between the primal problem and dual problem? Why?

3. (15 points)
   Find the Lagrangian dual problem of the following problem:

$$\begin{aligned} Minimize \quad & f(x) = x_1^2 - x_2 \\ \text{s.t.} \quad & h(x) = x_2 - \frac{\sqrt{2}}{2} = 0 \\ & g(x) = x_1^2 + x_2^2 \leq 1 \\ & x \in X = E^2 \end{aligned}$$

   Is there any duality gap between the primal problem and dual problem? Why?

4. (15 points)
   Find the Lagrangian dual problem of the following problem:

$$\begin{aligned} Minimize \quad & f(x) = x_1^2 - x_2 \\ \text{s.t.} \quad & h(x) = x_2 - \frac{\sqrt{2}}{2} = 0 \\ & x \in X = \{x \in E^2 \mid x_1^2 + x_2^2 \leq 1\} \end{aligned}$$

   Is there any duality gap between the primal problem and dual problem? Why?

5. (15 points)
   Find the Lagrangian dual problem of the following problem:

$$\begin{aligned} Minimize \quad & f(x) = x_1^2 - x_2 \\ \text{s.t.} \quad & g(x) = x_1^2 + x_2^2 \leq 1 \end{aligned}$$

$$x \in X = \{x \in E^2 \mid x_2 - \frac{\sqrt{2}}{2} = 0\}$$

   Is there any duality gap between the primal problem and dual problem? Why?

6. (5 points)
   Putting the results of Problems 3, 4 and 5 together, what can you say about Largrangian dual problem corresponding to a given constrained nonlinear optimization problem?

7. (15 points)
   Find the Lagrangian dual problem of the following linear programming problem:

$$\begin{aligned} Maximize \quad & f(x) = c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in E_+^n \end{aligned}$$

$$where\ c \in E^n, A \in E^m \times E^n, b \in E^m.$$

8. (5 points)
   For a constrained nonlinear optimization problem, what are the advantages of solving its Lagrangian dual problem, if there exists no duality gap.