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Convex Analysis and Duality over Discrete Domains

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Abstract The aim of this paper is to establish a fundamental theory of convex analysis for the sets and functions over a discrete domain. By introducing conjugate/biconjugate functions and a discrete duality notion for the cones over discrete domains, we study duals of optimization problems whose decision parameters are integers. In particular, we construct duality theory for integer linear programming, provide a discrete version of Slater's condition that implies the strong duality and discuss the relationship between integrality and discrete convexity.

Keywords Discrete convex analysis · Discrete Lagrangian duality · Discrete Slater's condition · Discrete strong duality · Integer programming · Integrality

Mathematics Subject Classification Primary 90C10 · Secondary 90C46

Contents

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1 Introduction

In recent decades, the idea of developing a discrete analogue of conventional convex analysis has become the motivation of many researches. Along this direction, discretely convex functions were introduced by Miller [1] in 1971 and integrally convex functions were defined by Favati and Tardella [2] in 1990. Subsequently, several types of convexity defined on discrete domains have been proposed by Murota and his coauthors. In particular, the M-convex functions, L-convex functions, $M^\natural$-convex functions and $L^\natural$-convex functions were introduced in [3–6], respectively. Inspired by the properties of convex and semi-strictly quasiconvex functions, Ui [7] defined $\mathcal{D}$-convex and semistrictly quasi-$\mathcal{D}$-convex functions and discussed the connections between $\mathcal{D}$-convexity and aforementioned convexity definitions for functions defined on discrete domains. Characterizations of convexity on crystallographical lattices have been introduced by Doignon [8]. The theory of discrete duality also took the prominent attention of researchers in the field of integer programming. A comprehensive theory on discrete duality can be referred to Nemhauser and Wolsey [9] and Murota [10]. We may also refer to [11] and references therein for Lagrangian dual method for solving general integer programming problems.

Motivated by the general approach of [8,12], this paper aims to develop a new framework that enables the use of conventional convex analysis on discrete domains. Different from the above-mentioned literature, we start with introducing convex sets on the integer domain $\mathbb{Z}^n$ and mixed-integer domain $\mathbb{Z}^n \times \mathbb{R}$ and define the discrete convex function as a function whose epigraph is convex in $\mathbb{Z}^n \times \mathbb{R}$. In our approach, the discrete restriction $f|_{\mathbb{Z}^n}$ of a convex function f on the real domain is convex on $\mathbb{Z}^n$. Conversely, every convex function on a discrete domain can be extended to a convex function on the real domain. Unlike other types of convex functions defined on discrete domains, our simple approach enables the use of already-existing results of conventional convex analysis to check the convexity of functions defined on a discrete domain. For a real-valued function $f : \mathbb{Z}^n \to \mathbb{R}$, we define the conjugate and biconjugate functions by

$$f^*(y) = \sup\left\{\langle x, y\rangle - f(x) : x \in \mathbb{Z}^n\right\} \quad \text{on } \mathbb{R}^n$$

and

$$f^{**}_{\mathbb{Z}^n}(x) = \sup\left\{\langle y, x\rangle - f^*(y) : y \in \mathbb{R}^n\right\} \quad \text{on } \mathbb{Z}^n.$$

By introducing the convex-closure $\text{ccl}_{\mathbb{Z}^n}(f)$ of a function f on a discrete domain, we prove a discrete analogue of the Fenchel–Moreau theorem stating that the second dual (dual of dual) of the function f satisfies that $f^{**}_{\mathbb{Z}^n} = \text{ccl}_{\mathbb{Z}^n}(f)$ on $\mathbb{Z}^n$. Unlike Murota's approach, our dual problem based the conjugate function f^* works on a continuous domain instead of on a discrete domain. We then demonstrate how the furnished results can be implemented to construct duality theory for integer programming problems. In particular, we provide a discrete analogue of Slater's condition that implies the strong duality for integer linear programming and show the essential role of convexity notion for the sets in a discrete domain. The analytical approach adopted in this study can be further extended to optimization on mixed-integer domains and quasiconvex analysis on discrete domains.

The rest of the paper is structured as follows: In Sects. 2 and 3, by introducing the concepts of convex-closure, convex-interior, convex-boundary of sets in a discrete domain, we study some topological and algebraic properties of convex and affine sets in a discrete domain. In Sect. 4, using the convexity notion for sets in a mixed-integer domain, we introduce the notion of convex functions on a discrete domain. Section 5 is devoted to the notion of convex-relative interiors. In Sects. 6–8, we further investigate the properties of convex functions on a discrete domain, introduce the concept of convex-lower semi-continuity, define conjugate/biconjugate functions and prove a discrete analogue of the Fenchel–Moreau theorem. In the last section, we show our duality theory for integer programming and provide some sufficient conditions for assuring the strong duality in integer linear programming.

Throughout this study, we denote by $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of positive integers, $\mathbb{Z}^n$ the n-dimensional Cartesian product of $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space. For a set $S \subset \mathbb{R}^n$, its closure is denoted by $\overline{S}$ or $\text{cl}_{\mathbb{R}^n}(S)$; its interior is denoted by $\overset{\circ}{S}$ or $\text{int}_{\mathbb{R}^n}(S)$; its boundary is denoted by ∂S or $\text{bdry}_{\mathbb{R}^n}(S)$; and its convex hull is denoted by $\text{conv}_{\mathbb{R}^n}(S)$, in the sense of the regular topology of $\mathbb{R}^n$ [13]. We also denote by $\lfloor . \rfloor$ and $\lceil . \rceil$ the greatest integer function (or floor function) and the least integer function (or ceiling function), respectively. All existing results of the literature are particularly noted.

2 Convex Sets in $\mathbb{Z}^n$

We first introduce the notion of convexity for sets in the discrete domain $\mathbb{Z}^n$ and discuss the properties of convex sets in $\mathbb{Z}^n$.

Definition 2.1 (*Convex set in* $\mathbb{Z}^n$) Let S be a subset of $\mathbb{Z}^n$. S is said to be convex in $\mathbb{Z}^n$ if and only if the expression $x + \lambda(y - x) \in S$ holds for all $x, y \in \text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n$ and $\lambda \in [0, 1]$ such that $x + \lambda(y - x) \in \mathbb{Z}^n$.

Since $S \subset \text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n$ for a set $S \subset \mathbb{Z}^n$, we obtain the following result:

Lemma 2.2 *A set $S \subset \mathbb{Z}^n$ is convex in $\mathbb{Z}^n$ if and only if*

$$S = \text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n. \quad \text{(Fig.1)}$$

Using Lemma 2.2, one may also define a convex set on a more general domain Λ^n which can turn into $\mathbb{Z}^n$ or product of $\mathbb{R}^m$ and $\mathbb{Z}^{n-m}$.

Definition 2.3 Let T_i, $i = 1, 2, \cdots, n$, be nonempty closed subsets of reals and Λ^n denote the product $T_1 \times T_2 \times \cdots \times T_n$. A set $S \subset \Lambda^n$ is convex in Λ^n if and only if

$$S = \text{conv}_{\mathbb{R}^n}(S) \cap \Lambda^n.$$

Lemma 2.4 *If A is a convex set in $\mathbb{R}^n$, then*

$$\text{conv}_{\mathbb{R}^n}\left(A \cap \mathbb{Z}^n\right) \cap \mathbb{Z}^n = A \cap \mathbb{Z}^n. \tag{2.1}$$

Proof Since $A \cap \mathbb{Z}^n \subset \text{conv}_{\mathbb{R}^n}(A \cap \mathbb{Z}^n)$, we have $A \cap \mathbb{Z}^n \subset \text{conv}_{\mathbb{R}^n}(A \cap \mathbb{Z}^n) \cap \mathbb{Z}^n$. When A is convex in $\mathbb{R}^n$, we further have

$$A \cap \mathbb{Z}^n = \text{conv}_{\mathbb{R}^n}(A) \cap \mathbb{Z}^n \supset \text{conv}_{\mathbb{R}^n}\left(A \cap \mathbb{Z}^n\right) \cap \mathbb{Z}^n.$$

This leads to the next result.

Theorem 2.5 *If A is a convex set in $\mathbb{R}^n$, then $A \cap \mathbb{Z}^n$ is convex in $\mathbb{Z}^n$.*

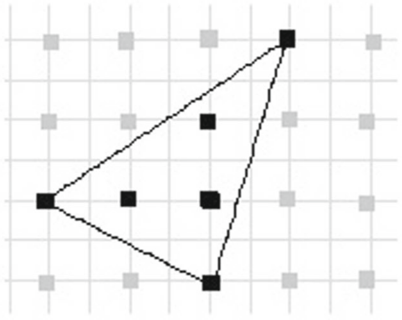
Definition 2.6 (*Convex hull in $\mathbb{Z}^n$*) The convex hull of a set S in $\mathbb{Z}^n$, denoted by $\text{conv}_{\mathbb{Z}^n}(S)$, is given by

$$\text{conv}_{\mathbb{Z}^n}(S) = \text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n.$$

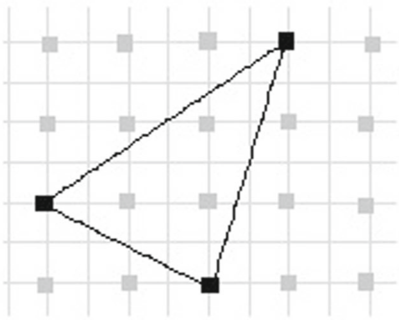
Using Theorem 2.5 and Definition 2.6, we have the following result:

Lemma 2.7 *Let S be a set in $\mathbb{Z}^n$. Then, we have*

1. *S is convex in $\mathbb{Z}^n$ if and only if there is a convex set $C \subset \mathbb{R}^n$ such that $S = C \cap \mathbb{Z}^n$.*
2. $\text{conv}_{\mathbb{Z}^n}(\text{conv}_{\mathbb{Z}^n}(S)) = \text{conv}_{\mathbb{Z}^n}(S)$.



A convex set in $\mathbb{Z}^2$



A non-convex set in $\mathbb{Z}^2$

Fig. 1 Convex and nonconvex sets in $\mathbb{Z}^2$

3. $\mathrm{conv}_{\mathbb{Z}^n}(S)$ *is the intersection of all convex sets in* $\mathbb{Z}^n$ *containing* S.
4. $\mathrm{conv}_{\mathbb{R}^n}(S) = \mathrm{conv}_{\mathbb{R}^n}(\mathrm{conv}_{\mathbb{Z}^n}(S))$.
5. *The intersection of an arbitrary collection of convex sets in* $\mathbb{Z}^n$ *is convex in* $\mathbb{Z}^n$.
6. *Let* S_1 *and* S_2 *be two convex sets in* $\mathbb{Z}^n$ *and* $\mathbb{Z}^m$, *respectively. Then,*

$$S_1 \times S_2 = \{(x, y) : x \in S_1 \quad \text{and} \quad y \in S_2\} \tag{2.2}$$

is a convex set in $\mathbb{Z}^n \times \mathbb{Z}^m$.
7. *For two sets* $S_1, S_2 \subset \mathbb{Z}^n$,

$$\mathrm{conv}_{\mathbb{Z}^n}(S_1) = \mathrm{conv}_{\mathbb{Z}^n}(S_2) \quad \text{if and only if} \quad \mathrm{conv}_{\mathbb{R}^n}(S_1) = \mathrm{conv}_{\mathbb{R}^n}(S_2).$$

2.1 Convex-Closure, Convex-Interior and Convex-Boundary of Sets in $\mathbb{Z}^n$

We now define more details of the sets in $\mathbb{Z}^n$.

Definition 2.8 For a set $S \subset \mathbb{Z}^n$, the convex-closure, convex-interior and convex-boundary of S in $\mathbb{Z}^n$, are defined by

$$\mathrm{ccl}_{\mathbb{Z}^n}(S) = \overline{\mathrm{conv}_{\mathbb{R}^n}(S)} \cap \mathbb{Z}^n,$$
$$\mathrm{cint}_{\mathbb{Z}^n}(S) = \overset{\circ}{\widehat{\mathrm{conv}_{\mathbb{R}^n}(S)}} \cap \mathbb{Z}^n,$$

and

$$\mathrm{cbdy}_{\mathbb{Z}^n}(S) = \partial\mathrm{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n,$$

respectively, where $\overline{\mathrm{conv}_{\mathbb{R}^n}(S)}$, $\overset{\circ}{\widehat{\mathrm{conv}_{\mathbb{R}^n}(S)}}$, and $\partial\mathrm{conv}_{\mathbb{R}^n}(S)$ indicate the closure, interior and boundary of the set $\mathrm{conv}_{\mathbb{R}^n}(S)$ in $\mathbb{R}^n$. Moreover, we say a set $S \subset \mathbb{Z}^n$ is integrally closed in $\mathbb{Z}^n$ if

$$S = \mathrm{ccl}_{\mathbb{Z}^n}(S). \tag{2.3}$$

For a convex set $S \subset \mathbb{Z}^n$, we always have

$$\mathrm{cint}_{\mathbb{Z}^n}(S) \subseteq S$$

and for any set $S \subset \mathbb{Z}^n$

$$S \subseteq \mathrm{ccl}_{\mathbb{Z}^n}(S).$$

Remark 2.9 The closeness of a convex set S in $\mathbb{Z}^n$ may not imply its integrally closeness in $\mathbb{Z}^n$. For example, the set

$$S = \{(x, 0) : x \in \mathbb{Z}\} \cup \{(0, 1)\}$$

is a closed set and it is convex in $\mathbb{Z}^2$, but it is not an integrally closed set in $\mathbb{Z}^2$ satisfying (2.3).

Lemma 2.10 *For a set S in $\mathbb{Z}^n$, its convex-closure $\mathrm{ccl}_{\mathbb{Z}^n}(S)$ and convex-interior $\mathrm{cint}_{\mathbb{Z}^n}(S)$ are both convex sets in $\mathbb{Z}^n$.*

2.2 Convex Cones in $\mathbb{Z}^n$

In this section, we define convex cones in $\mathbb{Z}^n$ and study their properties.

Definition 2.11 (*Cone in $\mathbb{Z}^n$*) A subset $K \subset \mathbb{Z}^n$ is a cone in $\mathbb{Z}^n$ if $\lambda x \in K$ holds for all $x \in K$ and $\lambda > 0$ such that $\lambda x \in \mathbb{Z}^n$.

Remark 2.12 A cone in $\mathbb{Z}^n$ is equal to the union of nonempty intersections of half-lines emanating from the origin with $\mathbb{Z}^n$. The origin itself may or may not be included.

Using the notion of convexity on $\mathbb{Z}^n$, we define a convex cone on $\mathbb{Z}^n$ as below.

Definition 2.13 (*Convex cone in $\mathbb{Z}^n$*) A cone in $\mathbb{Z}^n$ is called a convex cone if it is a convex set in $\mathbb{Z}^n$.

Definition 2.14 (*Conical combination in $\mathbb{Z}^n$*) An element $x \in \mathbb{Z}^n$ is called a conical combination of the elements $x^1, x^2, \cdots, x^k \in \mathbb{Z}^n$ if

$$x = \sum_{i=1}^{k} \lambda_i x^i \quad \text{for some} \quad \lambda_i \geqslant 0, \quad 1 \leqslant i \leqslant k.$$

An element $x \in \mathbb{Z}^n$ is called a strictly conical combination of the elements $x^1, x^2, \cdots, x^k \in \mathbb{Z}^n$ if

$$x = \sum_{i=1}^{k} \lambda_i x^i \quad \text{for some} \quad \lambda_i > 0, \quad 1 \leqslant i \leqslant k.$$

The following results follow from the definition.

Lemma 2.15 *A set $K \subset \mathbb{Z}^n$ is a convex cone in $\mathbb{Z}^n$ if and only if there is a convex cone $\widetilde{K}$ in $\mathbb{R}^n$ such that $K = \widetilde{K} \cap \mathbb{Z}^n$.*

Proof The proof of the sufficiency part is clear from Lemma 2.7. For the necessity part, assume that K is a convex cone in $\mathbb{Z}^n$. From [14, Corollary 2.6.3], the set

$$\widetilde{K} = \{\lambda x : \lambda > 0 \quad \text{and} \quad x \in \mathrm{conv}_{\mathbb{R}^n}(K)\}$$

is a convex cone in $\mathbb{R}^n$. It suffices to show that $K = \widetilde{K} \cap \mathbb{Z}^n$. Convexity of K yields $K \subset \widetilde{K} \cap \mathbb{Z}^n$. To show that $K \supset \widetilde{K} \cap \mathbb{Z}^n$, we assume the existence of an $x \in (\widetilde{K} \cap \mathbb{Z}^n)\backslash K$. Then,

$$x = \alpha \left(\sum_{i=1}^{k} \lambda_i x^i \right)$$

for some $\alpha > 0$, $\lambda_i \geqslant 0$ and $x^i \in K$, $i = 1, 2, \cdots, k$, such that $\sum_{i=1}^{k} \lambda_i = 1$.

If $\alpha \geqslant 1$, then we must have $\alpha x^i \in \mathrm{conv}_{\mathbb{R}^n}(K)$ for all $i = 1, 2, \cdots, k$ and $x^i \in K$ since $mx^i \in K$ for $m := \min\{m \in \mathbb{Z} : m \geqslant \alpha\}$. This along with convexity of K implies

$$x = \sum_{i=1}^{k} \lambda_i \left(\alpha x^i\right) \in \mathrm{conv}_{\mathbb{R}^n}(K) \cap \mathbb{Z}^n = K,$$

which is a contradiction.

If $0 < \alpha < 1$, then we can choose an integer $\widehat{m} := \min\{m \in \mathbb{Z} : m \geqslant \frac{1}{\alpha}\}$ for which we have $\widehat{m}x \in \widetilde{K} \cap \mathbb{Z}^n$ since $\widetilde{K}$ is a cone in $\mathbb{R}^n$. Moreover, we have $\widehat{m}x \notin K$ since $x \notin K$ and K is a cone in $\mathbb{Z}^n$. This leads to a contradiction since $\widehat{m}\alpha \geqslant 1$ and

$$\widehat{m}x = \sum_{i=1}^{k} \lambda_i \left(\widehat{m}\alpha x^i\right) \in \mathrm{conv}_{\mathbb{R}^n}(K) \cap \mathbb{Z}^n = K.$$

This completes the proof.

Observe that $K = \{(x, 0) : x = 1, 2, \cdots\}$ is a convex cone with

$$\widetilde{K} = \{(x, 0) : x > 0\} \supset \mathrm{conv}_{\mathbb{R}^n}(K) = \{(x, 0) : x \geqslant 1\}.$$

Thus, $\mathrm{conv}_{\mathbb{R}^n}(K) = \widetilde{K}$ is not true in general. In the following we provide a sufficient condition so that $\mathrm{conv}_{\mathbb{R}^n}(K) = \widetilde{K}$ holds.

Lemma 2.16 *Let $K \subset \mathbb{Z}^n$ be a convex cone in $\mathbb{Z}^n$ including the zero vector $\mathbf{0} \in \mathbb{Z}^n$ and let $\widetilde{K}$ be defined by*

$$\widetilde{K} := \{\lambda x : \lambda > 0 \quad \text{and} \quad x \in \mathrm{conv}_{\mathbb{R}^n}(K)\}.$$

Then

$$\widetilde{K} = \mathrm{conv}_{\mathbb{R}^n}(K), \tag{2.4}$$

and hence,

$$\mathrm{ccl}_{\mathbb{Z}^n}(K) = \mathrm{cl}_{\mathbb{R}^n}(\widetilde{K}) \cap \mathbb{Z}^n. \tag{2.5}$$

Proof We need to show that

$$\{\lambda x : \lambda > 0 \quad \text{and} \quad x \in \mathrm{conv}_{\mathbb{R}^n}(K)\} \subset \mathrm{conv}_{\mathbb{R}^n}(K).$$

If $x \in \widetilde{K}$, then

$$x = \alpha \left(\sum_{i=1}^{k} \lambda_i x^i\right)$$

for some $\alpha > 0$, $\lambda_i \geqslant 0$ and $x^i \in K$, $i = 1, 2, \cdots, k$, such that $\sum_{i=1}^{k} \lambda_i = 1$.

If $\alpha \geqslant 1$, then we must have $\alpha x^i \in \text{conv}_{\mathbb{R}^n}(K)$ for all $i = 1, 2, \cdots, k$ and $x^i \in K$ since $mx^i \in K$ for $m := \min\{m \in \mathbb{Z} : m \geqslant \alpha\}$. This implies

$$x = \sum_{i=1}^{k} \lambda_i \left(\alpha x^i\right) \in \text{conv}_{\mathbb{R}^n}(K).$$

If $0 < \alpha < 1$, then we have

$$\alpha x^i = \alpha x^i + (1-\alpha)\mathbf{0} \in \text{conv}_{\mathbb{R}^n}(K) \quad \text{for all} \quad x^i \in K$$

since $\mathbf{0} \in K$. Consequently, we have (2.4), and hence, $\text{ccl}_{\mathbb{Z}^n}(K) = \text{cl}_{\mathbb{R}^n}(\widetilde{K}) \cap \mathbb{Z}^n$. This completes the proof.

Corollary 2.17 *A set $K \subset \mathbb{Z}^n$ is a convex cone in $\mathbb{Z}^n$ if and only if it contains all strictly conical combinations of its elements.*

Proof From [15, Lemma 1.5], we know the set $\widetilde{K} = \{\lambda x : \lambda > 0 \text{ and } x \in \text{conv}_{\mathbb{R}^n}(K)\}$ is a convex cone in $\mathbb{R}^n$ containing all strictly conical combinations of its elements. The proof then follows from Lemma 2.15.

Corollary 2.18 *The intersection of an arbitrary collection of convex cones in $\mathbb{Z}^n$ is a convex cone in $\mathbb{Z}^n$.*

Proof The proof follows [14, Theorem 2.5] and Lemma 2.15.

Theorem 2.19 *If S is a convex set in $\mathbb{Z}^n$, then*

$$K = \{\lambda x : \lambda > 0 \quad \text{and} \quad x \in \text{conv}_{\mathbb{R}^n}(S)\} \cap \mathbb{Z}^n$$

is the smallest convex cone in $\mathbb{Z}^n$ containing S.

Proof If S is convex in $\mathbb{Z}^n$, then $S = \text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n$. By [14, Corollary 2.6.3], we know that the smallest convex cone in $\mathbb{R}^n$ containing $\text{conv}_{\mathbb{R}^n}(S)$ is the set $\widetilde{K}$ given by

$$\widetilde{K} = \{\lambda x : \lambda > 0 \quad \text{and} \quad x \in \text{conv}_{\mathbb{R}^n}(S)\}. \tag{2.6}$$

This fact and Lemma 2.15 imply that the smallest convex cone in $\mathbb{Z}^n$ containing the set $S = \text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n$ is $\widetilde{K} \cap \mathbb{Z}^n$. This completes the proof.

Definition 2.20 (*Dual cone*) Let K be a cone in $\mathbb{Z}^n$. The set

$$K^* = \left\{y \in \mathbb{R}^n : \langle x, y\rangle \geqslant 0 \quad \text{for all} \quad x \in K\right\}$$

is called the dual cone of K, where $\langle, \rangle$ denotes the inner product on $\mathbb{R}^n$.

Actually, the dual cone of a set in $\mathbb{Z}^n$ is defined in a conventional manner as the dual cone of a set in $\mathbb{R}^n$.

Example 2.21 Let $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. The dual cone O_1^* of the first orthant

$$O_1 = \{(x, y) : x, y \in \mathbb{Z} \quad \text{and} \quad x, y \geqslant 0\}$$

in $\mathbb{Z}^2$ is given by

$$O_1^* = \{(x, y) : x, y \in [0, \infty)\}.$$

Observe that O_1^* is the first orthant in $\mathbb{R}^2$.

With the linearity and continuity of the inner product operator $\langle , \rangle$ the next two results follow.

Lemma 2.22 *For any cone K in $\mathbb{Z}^n$, the dual cone K^* is a convex and closed cone in $\mathbb{R}^n$.*

Corollary 2.23 *If K_1 and K_2 are two cones in $\mathbb{Z}^n$ satisfying $K_1 \subseteq K_2$, then $K_2^* \subseteq K_1^*$.*

Definition 2.24 (*Second dual of a cone in $\mathbb{Z}^n$*) Let K be a cone in $\mathbb{Z}^n$, the set

$$K_{\mathbb{Z}^n}^{**} = \left\{x \in \mathbb{Z}^n : \langle x, y\rangle \geqslant 0 \quad \text{for all} \quad y \in K^*\right\}$$

is the second dual cone of K in $\mathbb{Z}^n$. In other words,

$$K_{\mathbb{Z}^n}^{**} = \left(K^*\right)^* \cap \mathbb{Z}^n, \tag{2.7}$$

where $(K^*)^*$ indicates the conventional dual of the real cone K^* in $\mathbb{R}^n$.

Theorem 2.25 *Let K be a cone in $\mathbb{Z}^n$ including the zero vector $\mathbf{0} \in \mathbb{Z}^n$. Then*

$$\text{ccl}_{\mathbb{Z}^n}(K) = K_{\mathbb{Z}^n}^{**}.$$

Consequently, if K is integrally closed in $\mathbb{Z}^n$ [i.e., $\text{ccl}_{\mathbb{Z}^n}(K) = K$] with $\mathbf{0} \in K$, then

$$K_{\mathbb{Z}^n}^{**} = K.$$

Proof For the cone K in $\mathbb{Z}^n$, the linearity of inner product yields

$$K^* = \left\{y \in \mathbb{R}^n : \langle x, y\rangle \geqslant 0 \quad \text{for all} \quad x \in \text{conv}_{\mathbb{R}^n}(K)\right\}.$$

Therefore, $y(\neq \mathbf{0}) \in K^*$ if and only if y is the normal vector of every closed half-space of the form

$$H_a^+ := \left\{x \in \mathbb{R}^n : \langle a, x\rangle \geqslant 0\right\}$$

including $\text{conv}_{\mathbb{R}^n}(K)$. By [14, Corollary 11.7.2], we have

$$\overline{\{\lambda x : \lambda > 0 \quad \text{and} \quad x \in \text{conv}_{\mathbb{R}^n}(K)\}} = \bigcap_{y \in K^*} H_y^+,$$

where H_y^+ indicates the half-space $\{x \in \mathbb{R}^n : \langle y, x\rangle \geqslant 0\}$ containing $\text{conv}_{\mathbb{R}^n}(K)$. Together with (2.5), we have

$$\begin{aligned} \text{ccl}_{\mathbb{Z}^n}(K) &= \left(\bigcap_{y\in K^*} H_y^+\right) \cap \mathbb{Z}^n \\ &= \left\{x \in \mathbb{R}^n : \langle y, x\rangle \geqslant 0 \quad \text{for all} \quad y \in K^*\right\} \cap \mathbb{Z}^n \\ &= \left(K^*\right)^* \cap \mathbb{Z}^n = K^{**}_{\mathbb{Z}^n}. \end{aligned}$$

This completes the proof.

Hereafter, we will define a pointed cone in $\mathbb{Z}^n$ by using the conventional definition [16, 2.4.2 Definition] of pointed cone C in $\mathbb{R}^n$. It should be mentioned that the authors in [17] and [16] defined the convex cone C in $\mathbb{R}^n$ as a convex set satisfying $\lambda C \subseteq C$ for all $\lambda \geqslant 0$. However, in our approach we regard the convex cone C in $\mathbb{R}^n$ to be the set satisfying $\lambda C \subseteq C$ for all $\lambda > 0$. This type of approach has also been adopted by many authors in conventional convex analysis (see for instance [14,15]). Bearing this fact in mind and using [16, 2.4.2 Definition], we can derive the following variant of definition of pointed cone in $\mathbb{R}^n$ which may not include the origin.

Definition 2.26 (*Pointed cone in* $\mathbb{R}^n$) A convex cone C in $\mathbb{R}^n$ is pointed if the set

$$\widetilde{C}_0 = \{\lambda x : \lambda \geqslant 0 \quad \text{and} \quad x \in C\}$$

satisfies the following

$$\widetilde{C}_0 \cap -\widetilde{C}_0 = \{\mathbf{0}\}.$$

Notice that when C is a closed and convex cone in $\mathbb{R}^n$, $C = \widetilde{C}_0$ is always true. We now define a pointed convex cone in $\mathbb{Z}^n$.

Definition 2.27 (*Pointed cone in* $\mathbb{Z}^n$) A convex cone K in $\mathbb{Z}^n$ is pointed if the set

$$\widetilde{K}_0 = \{\lambda x : \lambda \geqslant 0 \quad \text{and} \quad x \in \text{conv}_{\mathbb{R}^n}(K)\}$$

satisfies that

$$\widetilde{K}_0 \cap -\widetilde{K}_0 = \{\mathbf{0}\}.$$

Example 2.28 Consider the convex cone $K = \{(s_1, s_2) : s_1, s_2 \in [1, \infty) \cap \mathbb{Z}\}$ in $\mathbb{Z}^2$. Then we obtain

$$\widetilde{K}_0 = ((0, \infty) \times (0, \infty)) \cup \{\mathbf{0}\}.$$

Since $\widetilde{K} \cap -\widetilde{K} = \{\mathbf{0}\}$ the given set K is a pointed cone in $\mathbb{Z}^2$.

In conventional convex analysis, a set A in $\mathbb{R}^n$ is said to be solid if it has a nonempty interior (see [16, 2.4.2 Definition]). Inspired by this definition we give the next definition.

Definition 2.29 (*Solid cone in* $\mathbb{Z}^n$) A convex cone K in $\mathbb{Z}^n$ is solid if $\mathrm{cint}_{\mathbb{Z}^n}(K) \neq \varnothing$.

The next result is commonly seen in convex analysis.

Theorem 2.30 [16, 2.4.3 Proposition] *If C is a closed convex cone in $\mathbb{R}^n$, then we have*

a. If C is solid, then its dual C^* is pointed,
b. If C is pointed, then its dual C^* is solid.

Thus C is solid (pointed) if and only if C^* is pointed (solid).

This theorem leads to the following result for cones over integer domains:

Theorem 2.31 *Let K be an integrally closed cone in $\mathbb{Z}^n$ and $\mathbf{0} \in K$. Then we have*

a. If K is solid in $\mathbb{Z}^n$, then its dual K^* is pointed in $\mathbb{R}^n$,
b. If K is pointed in $\mathbb{Z}^n$, then its dual K^* is solid in $\mathbb{R}^n$.

Proof Let $\widetilde{K} = \{\lambda x : \lambda > 0 \text{ and } x \in \mathrm{conv}_{\mathbb{R}^n}(K)\}$. If $K \subset \mathbb{Z}^n$ is a integrally closed cone including zero vector $\mathbf{0} \in \mathbb{Z}^n$, then (2.4) yields

$$\mathrm{cl}_{\mathbb{R}^n}\left(\widetilde{K}\right) = \overline{\mathrm{conv}_{\mathbb{R}^n}(K)}. \tag{2.8}$$

Since $\overline{\mathrm{conv}_{\mathbb{R}^n}(K)} = \mathrm{conv}_{\mathbb{R}^n}(\mathrm{ccl}_{\mathbb{Z}^n}(K))$ (see Proposition 5.10 in Sect. 5), we get from $K = \mathrm{ccl}_{\mathbb{Z}^n}(K)$, (2.4), and (2.8) that $\mathrm{cl}_{\mathbb{R}^n}(\widetilde{K}) = \widetilde{K}$.
Observe that

$$\mathrm{cint}_{\mathbb{Z}^n}(K) \subseteq \mathrm{int}\,(\mathrm{conv}_{\mathbb{R}^n}(K)) \subseteq \mathrm{int}\left(\mathrm{cl}_{\mathbb{R}^n}\left(\widetilde{K}\right)\right).$$

If the cone K is solid [i.e., if $\mathrm{cint}_{\mathbb{Z}^n}(K) \neq \varnothing$], then the closed and convex cone $\mathrm{cl}_{\mathbb{R}^n}(\widetilde{K})$ in $\mathbb{R}^n$ is solid. This along with preceding theorem implies that the dual cone $(\mathrm{cl}_{\mathbb{R}^n}(\widetilde{K}))^*$ is pointed. By continuity and linearity of the inner product, we obtain

$$\left(\mathrm{cl}_{\mathbb{R}^n}\left(\widetilde{K}\right)\right)^* = \widetilde{K}^* = K^*.$$

This shows that K^* is pointed in $\mathbb{R}^n$. Conversely, if K is pointed, then by definition the convex cone $\widetilde{K}$ is pointed in $\mathbb{R}^n$. Since $\widetilde{K}$ is closed and convex, by the preceding theorem $(\widetilde{K})^*$ is solid. Thus, $K^* = (\widetilde{K})^*$ is solid in $\mathbb{R}^n$. This completes the proof.

3 Affine Sets in $\mathbb{Z}^n$

In this section, the concept and properties of affine sets in $\mathbb{Z}^n$ are introduced. Recall that $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and the set of positive integers, respectively. Also recall the next definition in convex analysis.

Definition 3.1 (*Affine combination and affine hull in* $\mathbb{R}^n$) For a set $S \subset \mathbb{R}^n$, an element $x \in \mathbb{R}^n$ is an affine combination of elements of S if and only if

$$x = \sum_{i=1}^{m} \lambda_i x^i$$

for some $m \in \mathbb{N}$, $x^1, x^2, \cdots, x^m \in S$, and $\lambda_i \in \mathbb{R}$ with $\sum_{i=1}^{m} \lambda_i = 1$.

The affine hull of S, denoted by $\text{aff}_{\mathbb{R}^n}(S)$, is the collection of all affine combinations of elements of S in $\mathbb{R}^n$.

Similarly, we define the affine hull in $\mathbb{Z}^n$.

Definition 3.2 (*Affine hull in* $\mathbb{Z}^n$) For a set $S \subset \mathbb{Z}^n$, an element x is an affine combination of elements of S in $\mathbb{Z}^n$ if and only if

$$x = \sum_{i=1}^{m} \lambda_i x^i$$

for some $m \in \mathbb{N}$, $x^1, x^2, \cdots, x^m \in S$, and $\lambda_i \in \mathbb{R}$ such that $\sum_{i=1}^{m} \lambda_i = 1$ and $\sum_{i=1}^{m} \lambda_i x^i \in \mathbb{Z}^n$.

The affine hull of S in $\mathbb{Z}^n$, denoted by $\text{aff}_{\mathbb{Z}^n}(S)$, is the collection of all affine combinations of elements of S in $\mathbb{Z}^n$.

The affine hull $\text{aff}_{\mathbb{Z}^n}(S)$ can alternatively be stated as follows:

Lemma 3.3 *For any set S in $\mathbb{Z}^n$*

$$\text{aff}_{\mathbb{Z}^n}(S) = \text{aff}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n. \tag{3.1}$$

Definition 3.4 (*Affine set in* $\mathbb{Z}^n$) For a set $S \subset \mathbb{Z}^n$, S is said to be affine in $\mathbb{Z}^n$ if and only if $x + \lambda(y - x) \in S$ for all $x, y \in \text{aff}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n$ and $\lambda \in \mathbb{R}$ such that $x + \lambda(y - x) \in \mathbb{Z}^n$.

This definition can alternatively be stated as below:

Lemma 3.5 *For a set $S \subset \mathbb{Z}$, S is affine in $\mathbb{Z}^n$ if and only if*

$$\text{aff}_{\mathbb{Z}^n}(S) = S. \tag{3.2}$$

Lemma 3.6 *If A is an affine set in $\mathbb{R}^n$, then*

$$\text{aff}_{\mathbb{R}^n}\left(A \cap \mathbb{Z}^n\right) \cap \mathbb{Z}^n = A \cap \mathbb{Z}^n. \tag{3.3}$$

Proof Since $A \cap \mathbb{Z}^n \subset \text{aff}_{\mathbb{R}^n}(A \cap \mathbb{Z}^n)$, we have $A \cap \mathbb{Z}^n \subset \text{aff}_{\mathbb{R}^n}(A \cap \mathbb{Z}^n) \cap \mathbb{Z}^n$. If A is affine in $\mathbb{R}^n$, then

$$\text{aff}_{\mathbb{R}^n}\left(A \cap \mathbb{Z}^n\right) \cap \mathbb{Z}^n \subset \text{aff}_{\mathbb{R}^n}(A) \cap \mathbb{Z}^n = A \cap \mathbb{Z}^n.$$

Substituting S for $A \cap \mathbb{Z}^n$ in (3.3) and using Lemma 3.5, we have the next result.

Theorem 3.7 *If A is an affine set in $\mathbb{R}^n$, then $A \cap \mathbb{Z}^n$ is an affine set in $\mathbb{Z}^n$.*

Next, we provide a characterization of the affine sets in $\mathbb{Z}^n$.

Corollary 3.8 *A set S in $\mathbb{Z}^n$ is affine if and only if there is an affine set A in $\mathbb{R}^n$ such that $S = A \cap \mathbb{Z}^n$.*

Proof Lemma 3.6 proves the necessity part. For the sufficiency part, let S be an affine set in $\mathbb{Z}^n$, then $A = \text{aff}_{\mathbb{R}^n}(S)$ can be given as the desired set satisfying $S = A \cap \mathbb{Z}^n$. The proof follows.

The next result is a direct consequence of Lemmas 3.3 and 3.6.

Corollary 3.9 *For any set $S \subset \mathbb{Z}^n$,*

$$\text{aff}_{\mathbb{Z}^n}(\text{aff}_{\mathbb{Z}^n}(S)) = \text{aff}_{\mathbb{R}^n}\left(\text{aff}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n\right) \cap \mathbb{Z}^n = \text{aff}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n = \text{aff}_{\mathbb{Z}^n}(S).$$

Thus, $\text{aff}_{\mathbb{Z}^n}(S)$ *is an affine set in $\mathbb{Z}^n$.*

Using (3.1) and the properties of the affine hull $\text{aff}_{\mathbb{R}^n}(S)$ in $\mathbb{R}^n$ (see [14, Section 1]), we have some properties of the affine $\text{aff}_{\mathbb{Z}^n}(S)$ hull in $\mathbb{Z}^n$ listed below.

Corollary 3.10 *For any set $S \subset \mathbb{Z}^n$, $\text{aff}_{\mathbb{Z}^n}(S)$ is the intersection of the collection of affine sets in $\mathbb{Z}^n$ containing S.*

Corollary 3.11 *For any set $S \subset \mathbb{Z}^n$, $\text{aff}_{\mathbb{Z}^n}(S)$ is the smallest affine set in $\mathbb{Z}^n$ containing S.*

Next, we give the relationship between $\text{aff}_{\mathbb{R}^n}(S)$ and $\text{aff}_{\mathbb{Z}^n}(S)$.

Corollary 3.12 *For any set $S \subset \mathbb{Z}^n$,*

$$\text{aff}_{\mathbb{R}^n}(S) = \text{aff}_{\mathbb{R}^n}\left(\text{aff}_{\mathbb{Z}^n}(S)\right).$$

Proof For $S \subset \mathbb{Z}^n$, since $S \subset \text{aff}_{\mathbb{Z}^n}(S)$ and $\text{aff}_{\mathbb{Z}^n}(S) \subset \text{aff}_{\mathbb{R}^n}(S)$, we know that $\text{aff}_{\mathbb{R}^n}(S) \subset \text{aff}_{\mathbb{R}^n}(\text{aff}_{\mathbb{Z}^n}(S))$ and

$$\text{aff}_{\mathbb{R}^n}(\text{aff}_{\mathbb{Z}^n}(S)) \subset \text{aff}_{\mathbb{R}^n}(\text{aff}_{\mathbb{R}^n}(S)) = \text{aff}_{\mathbb{R}^n}(S),$$

respectively. The proof follows.

Corollary 3.13 *For any $S_1, S_2 \subset \mathbb{Z}^n$,*

$$\text{aff}_{\mathbb{Z}^n}(S_1) = \text{aff}_{\mathbb{Z}^n}(S_2) \quad \text{if and only if} \quad \text{aff}_{\mathbb{R}^n}(S_1) = \text{aff}_{\mathbb{R}^n}(S_2). \tag{3.4}$$

Proof It is a direct consequence of Corollary 3.12 and (3.1).

4 Convex Functions on $\mathbb{Z}^n$

In this section, we define extended real-valued convex functions on $\mathbb{Z}^n$ and investigate their basic properties.

Definition 4.1 (*Epigraph of a function on* $\mathbb{Z}^n$) Let $S \subset \mathbb{Z}^n$ and $f : S \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued function defined on S. Its epigraph, denoted by epi(f) is defined by

$$\text{epi}(f) = \{(x, y) \in S \times \mathbb{R} : y \geqslant f(x)\}.$$

The inner epigraph is defined by

$$\widetilde{\text{epi}(f)} = \{(x, y) \in S \times \mathbb{R} : y > f(x)\}.$$

Remark 4.2 When $S = \mathbb{Z}^n$,

$$\text{epi}(f) = \left\{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : y \geqslant f(x)\right\}.$$

We call the function f in the above definition a "function on discrete domain" or "discrete function" in this paper. Notice that the epigraph of a discrete function f is a set in a mixed domain of $\mathbb{Z}^n \times \mathbb{R}$.

Definition 4.3 (*Effective domain of a function on discrete domain*) Let f be a function on a discrete domain $S \subset \mathbb{Z}^n$. Its effective domain, denoted by dom(f), is defined to be the projection of the epigraph of f onto $\mathbb{Z}^n$, i.e.,

$$\begin{aligned} \text{dom}(f) &= \pi_{\mathbb{Z}^n}(\text{epi}(f)) \\ &= \{x \in S : \exists\, y \in \mathbb{R} \text{ s.t. } (x, y) \in \text{epi}\,(f)\} \\ &= \{x \in S : f(x) < +\infty\}, \end{aligned}$$

where $\pi_{\mathbb{Z}^n}$ denotes the projection from $\mathbb{Z}^n \times \mathbb{R}$ to $\mathbb{Z}^n$. Moreover, the dimension of the unique subspace parallel to aff$_{\mathbb{R}^n}$(dom(f)) is called the dimension of dom(f) (see [14, Theorem 1.2]).

Remark 4.4 When $S = \mathbb{Z}^n$, then

$$\text{dom}(f) = \left\{x \in \mathbb{Z}^n : f(x) < +\infty\right\} = \pi_1(\text{epi}(f)), \tag{4.1}$$

where π_1 denotes the projection from $\mathbb{R}^n \times \mathbb{R}$ onto $\mathbb{R}^n$. It is useful to represent a function f on $\mathbb{Z}^n$ by the following relation:

$$f(x) = \inf\{y \in \mathbb{R} : (x, y) \in \text{epi}\,(f)\}. \tag{4.2}$$

Here, we take $\inf \varnothing = +\infty$ which happens only at $x \in \mathbb{Z}^n$ but $x \notin \text{dom}(f)$.

Using Definition 2.3, we define a convex function f on $\mathbb{Z}^n$ in terms of convexity of epi(f) in $\mathbb{Z}^n \times \mathbb{R}$.

Definition 4.5 (*Convex/concave/affine function on* $\mathbb{Z}^n$) Let f be a function on a discrete domain $S \subset \mathbb{Z}^n$. We say f is a convex function on $\mathbb{Z}^n$ if and only if its epigraph is a convex set in $\mathbb{Z}^n \times \mathbb{R}$, i.e.,

$$\text{epi}\,(f) = \text{conv}_{\mathbb{R}^{n+1}}\,(\text{epi}\,(f)) \cap \left(\mathbb{Z}^n \times \mathbb{R}\right). \tag{4.3}$$

A function g on a discrete domain $S \subset \mathbb{Z}^n$ is a concave function if the function $-g$ is a convex function on $\mathbb{Z}^n$. A function h on a discrete domain $S \subset \mathbb{Z}^n$ is an affine function if it is finite, convex and concave on $\mathbb{Z}^n$.

One may easily verify the next result by using Definitions 4.3 and 4.5.

Lemma 4.6 *If f is a convex function on $\mathbb{Z}^n$, then* dom (f) *is a convex set in $\mathbb{Z}^n$.*

Corollary 4.7 *If f is a convex function on $\mathbb{Z}^n$, then*

$$\pi_1\,(\text{epi}\,(f)) = \pi_1\left(\text{conv}_{\mathbb{R}^{n+1}}\,(\text{epi}\,(f))\right) \cap \mathbb{Z}^n,$$

where π_1 is the projection mapping from $\mathbb{R}^n \times \mathbb{R}$ onto $\mathbb{R}^n$.

Proof The proof follows from (4.3) and the equality

$$\pi_1\left[\text{conv}_{\mathbb{R}^{n+1}}\,(\text{epi}\,(f)) \cap \left(\mathbb{Z}^n \times \mathbb{R}\right)\right] = \pi_1\left(\text{conv}_{\mathbb{R}^{n+1}}\,(\text{epi}\,(f))\right) \cap \mathbb{Z}^n.$$

Corollary 4.8 *Let A be a convex set in $\mathbb{R}^n$ such that $A \cap \mathbb{Z}^n$ is nonempty. If g is a convex function on a convex set A, then the restriction $f = g|_{A\cap\mathbb{Z}^n}$ of g to $A \cap \mathbb{Z}^n$ is a convex function on $A \cap \mathbb{Z}^n$.*

Proof If g is a convex function on a convex set A in $\mathbb{R}^n$, then epi (g) is a convex set in $\mathbb{R}^{n+1}$. As we did in Lemma 2.7, we get

$$\text{epi}\left(g|_{A\cap\mathbb{Z}^n}\right) = \text{epi}\,(g) \cap (\left(A \cap \mathbb{Z}^n\right) \times \mathbb{R}),$$

which shows convexity of epi $\left(g|_{A\cap\mathbb{Z}^n}\right)$ in $\mathbb{Z}^n \times \mathbb{R}$). This completes the proof.

Remark 4.9 A convex function f on $S \subset \mathbb{Z}^n$ can always be extended to a convex function on whole $\mathbb{Z}^n$ by setting $f(x) = +\infty$ for $x \notin S$. By a "convex function", we shall henceforth always mean a "convex function with possibly infinite values which is defined throughout $\mathbb{Z}^n$",unless otherwise specified. The rules we adopt are listed below:

$$\begin{gathered}
\alpha + \infty = \infty + \alpha = \infty \quad \text{for} \quad -\infty < \alpha \leqslant +\infty, \\
\alpha - \infty = -\infty + \alpha = -\infty \quad \text{for} \quad -\infty \leqslant \alpha < +\infty, \\
\alpha\infty = \infty\alpha = \infty, \quad \alpha(-\infty) = (-\infty)\alpha = -\infty \quad \text{for} \quad 0 < \alpha \leqslant +\infty, \\
\alpha\infty = \infty\alpha = -\infty, \quad \alpha(-\infty) = (-\infty)\alpha = \infty \quad \text{for} \quad -\infty \leqslant \alpha < 0, \\
0\infty = \infty 0 = 0 = 0(-\infty) = (-\infty)0, \quad -(-\infty) = \infty, \\
\inf \varnothing = +\infty, \quad \sup \varnothing = -\infty.
\end{gathered}$$

Note that $\infty - \infty$ and $-\infty + \infty$ are undefined and to be avoided.

Definition 4.10 (*Proper/improper convex function on* $\mathbb{Z}^n$) Let f be a convex function on $\mathbb{Z}^n$. We say f is a proper function if its epigraph is nonempty and contains no vertical lines, i.e., $f(x) < +\infty$ for at least one $x \in \mathbb{Z}^n$ and $f(x) > -\infty$ for all $x \in \mathbb{Z}^n$. A convex function on $\mathbb{Z}^n$ which is not proper is called improper.

Lemma 4.11 *A convex function* f *on* $\mathbb{Z}^n$ *is proper if and only if the convex set* $S = \text{dom}(f) \subset \mathbb{Z}^n$ *is nonempty and the restriction of* f *to* S *is finite, i.e.,* $-\infty < f(x) < +\infty$ *for all* $x \in S$.

Lemma 4.12 *Let* f *be a finite convex function on a convex set* $S \subset \mathbb{Z}^n$. *Then the function*

$$F(x) = \begin{cases} f(x), & \text{if } x \in S, \\ +\infty, & \text{if } x \notin S \end{cases}$$

is a proper convex function on $\mathbb{Z}^n$.

Example 4.13 Let f be a function on $\mathbb{Z}$ given by

$$f(x) = \begin{cases} -\infty, & \text{if } |x| < 1, \\ 0, & \text{if } |x| = 1, \\ \infty, & \text{if } |x| > 1 \end{cases}$$

is an improper convex function on $\mathbb{Z}$.

An alternative criteria for convexity of a function f on $S \subset \mathbb{Z}^n$ can be given as follows.

Theorem 4.14 *A function* f *on a convex set* $S \subset \mathbb{Z}^n$ *is convex in* $\mathbb{Z}^n$ *if and only if*

$$f\left(\sum_{i=1}^{m} \lambda_i x^i\right) \leqslant \sum_{i=1}^{m} \lambda_i y^i \tag{4.4}$$

for all $m \in \mathbb{N}$ *and all* $x^i \in S$, $y^i \in \mathbb{R}$, *and* $\lambda_i \in [0, 1]$, $i = 1, 2, \cdots, m$, *such that* $\sum_{i=1}^{m} \lambda_i = 1$, $\sum_{i=1}^{m} \lambda_i x^i \in \mathbb{Z}^n$, *and* $f(x^i) \leqslant y^i$.

Condition (4.4) can be expressed in different ways. The next two variants would be especially useful.

Theorem 4.15 *Let* f *be an extended real-valued function defined from a convex set* $S \subset \mathbb{Z}^n$ *to* $(-\infty, \infty]$. f *is convex on* S *if and only if*

$$f\left(\sum_{i=1}^{m} \lambda_i x^i\right) \leqslant \sum_{i=1}^{m} \lambda_i f\left(x^i\right)$$

for all $m \in \mathbb{N}$ *and all* $x^i \in S$ *and* $\lambda_i \in [0, 1]$, $i = 1, 2, \cdots, m$, *such that* $\sum_{i=1}^{m} \lambda_i = 1$ *and* $\sum_{i=1}^{m} \lambda_i x^i \in \mathbb{Z}^n$.

Corollary 4.16 *If f_1 and f_2 are proper convex functions on $\mathbb{Z}^n$, then $f_1 + f_2$ is convex on $\mathbb{Z}^n$.*

Proof This is a direct consequence of Theorem 4.15.

Remark 4.17 Notice that for proper convex functions f_1 and f_2, we have $(f_1 + f_2)(x) < \infty$ if and only if $f_1(x) < \infty$ and $f_2(x) < \infty$. Thus, the effective domain $\operatorname{dom}(f_1 + f_2)$ of $f_1 + f_2$, satisfying that $\operatorname{dom}(f_1 + f_2) = \operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$, may become empty. In this case, $f_1 + f_2$ becomes improper.

Theorem 4.18 *Let f be an extended real-valued function defined from a convex set $S \subset \mathbb{Z}^n$ to $[-\infty, \infty]$. f is convex on S if and only if*

$$f\left(\sum_{i=1}^{m} \lambda_i x^i\right) < \sum_{i=1}^{m} \lambda_i y^i$$

for all $m \in \mathbb{N}$ and all $x^i \in S$, $y^i \in \mathbb{R}$ and $\lambda_i \in (0, 1)$, $i = 1, 2, \cdots, m$, such that $\sum_{i=1}^{m} \lambda_i = 1$, $\sum_{i=1}^{m} \lambda_i x^i \in \mathbb{Z}^n$ and $f(x^i) < y^i$.

Corollary 4.19 *A function $f : \mathbb{Z}^n \to [-\infty, \infty]$ is convex if and only if its inner epigraph*

$$\widetilde{\operatorname{epi}(f)} := \left\{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : f(x) < y\right\} \tag{4.5}$$

is convex in $\mathbb{Z}^n \times \mathbb{R}$.

Corollary 4.8 and Theorem 4.15 lead to the next result.

Corollary 4.20 1. *Every function of the form*

$$f(x) = \langle x, a\rangle + \alpha,\ a \in \mathbb{R}^n \quad \text{and} \quad \alpha \in \mathbb{R}$$

is convex on $\mathbb{Z}^n$.

2. *A quadratic function*

$$f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle x, a\rangle + \alpha,$$

where Q is a symmetric $n \times n$ matrix, is convex on $\mathbb{Z}^n$ if and only if Q is positive semi-definite, i.e.,

$$\langle z, Qz\rangle \geqslant 0 \quad \text{for every} \quad z \in \mathbb{R}^n.$$

3. *The indicator function*

$$\delta(x : S) = \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{if } x \notin S \end{cases}$$

of a set S in $\mathbb{Z}^n$ is convex if and only if S is convex in $\mathbb{Z}^n$.

Theorem 4.21 *For any convex function f in $\mathbb{Z}^n$ and any $\alpha \in [-\infty, +\infty]$, the level sets $\{x \in \mathbb{Z}^n : f(x) < \alpha\}$ and $\{x \in \mathbb{Z}^n : f(x) \leqslant \alpha\}$ are convex in $\mathbb{Z}^n$.*

Proof Convexity of the set $\{x \in \mathbb{Z}^n : f(x) < \alpha\}$ follows from Theorem 4.18, with $y^i = \alpha$. The convexity of the set $\{x \in \mathbb{Z}^n : f(x) \leqslant \alpha\}$ comes from Lemma 2.7 (5) which implies the convexity of the set

$$\bigcap_{y>\alpha} \left\{x \in \mathbb{Z}^n : f(x) < y\right\},$$

which coincides with the set $\{x \in \mathbb{Z}^n : f(x) \leqslant \alpha\}$.

Using Lemma 2.7 (5), we have the next result.

Corollary 4.22 *Let I be an index set, f_i be a convex function on $\mathbb{Z}^n$ and $\alpha_i \in \mathbb{R}$ for each $i \in I$, then*

$$C = \bigcap_{i \in I} \left\{x \in \mathbb{Z}^n : f_i(x) \leqslant \alpha_i\right\}$$

is a convex set in $\mathbb{Z}^n$.

Corollary 4.23 *Let I be an index set and f_i be a convex function on $\mathbb{Z}^n$ for each $i \in I$, then*

$$f(x) = \sup\{f_i(x) : i \in I\} \tag{4.6}$$

is a convex function on $\mathbb{Z}^n$.

Proof If f is given by (4.6), then the epigraph of f coincides with $\cap_{i \in I} \text{epi}(f_i)$ which is a convex set in $\mathbb{Z}^n \times \mathbb{R}$. The proof follows.

For the conventional convex analysis, we have the following result:

Theorem 4.24 ([14, Theorem 3.4]) *Let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then,*

$$AC = \{Ax : x \in C\}$$

is a convex set in $\mathbb{R}^m$ for every convex set C in $\mathbb{R}^n$, and

$$A^{-1}D = \left\{x \in \mathbb{R}^n : Ax \in D\right\}$$

is a convex set in $\mathbb{R}^n$ for every convex set D in $\mathbb{R}^m$.

The next example shows that discrete convexity may not be preserved under linear mappings.

Example 4.25 Let $\pi_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the projection operator defined by $\pi_1(x_1, x_2) = x_1$. Obviously, $\pi_1 S = \{1, 3\}$ is not a convex set in $\mathbb{Z}$ while $S = \{(1, 1), (3, 2)\}$ is a convex set in $\mathbb{Z} \times \mathbb{Z}$.

Lemma 4.26 *Let $S \subset \mathbb{Z}^n$ be convex in $\mathbb{Z}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation satisfying*

$$A\left(\text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n\right) = A\text{conv}_{\mathbb{R}^m}(S) \cap \mathbb{Z}^m.$$

Then the set AS is convex in $\mathbb{Z}^m$.

Proof Since S is convex in $\mathbb{Z}^n$ and A is linear, we have

$$\begin{aligned} AS &= A\left(\text{conv}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n\right) \\ &= A\text{conv}_{\mathbb{R}^m}(S) \cap \mathbb{Z}^m \\ &= \text{conv}_{\mathbb{R}^m}(AS) \cap \mathbb{Z}^m \\ &= \text{conv}_{\mathbb{Z}^m}(AS). \end{aligned}$$

Lemma 4.27 *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with $A\mathbb{Z}^n = \mathbb{Z}^m$. If D is a convex set in $\mathbb{Z}^m$, then $A^{-1}D$ is a convex set in $\mathbb{Z}^n$.*

Proof Let D be a convex set in $\mathbb{Z}^m$. Suppose that $x^1, x^2, \cdots, x^r$ are the elements of $A^{-1}D$ and $\lambda_i \in [0, 1]$, $i = 1, 2, \cdots, r$, are the scalars such that $\sum_{i=1}^r \lambda_i = 1$ and $\sum_{i=1}^r \lambda_i x^i \in \mathbb{Z}^n$. Since $A\mathbb{Z}^n = \mathbb{Z}^m$, we know that

$$A\left(\sum_{i=1}^r \lambda_i x^i\right) = \sum_{i=1}^r \lambda_i A x^i \in \mathbb{Z}^m.$$

By the convexity of D, we have

$$A\left(\sum_{i=1}^r \lambda_i x^i\right) = \sum_{i=1}^r \lambda_i A x^i \in D,$$

which implies that

$$\sum_{i=1}^r \lambda_i x^i \in A^{-1}D.$$

Consequently, $A^{-1}D$ is a convex set in $\mathbb{Z}^n$.

5 Convex-Relative Interiors of sets in $\mathbb{Z}^n$

Let $C \subset \mathbb{R}^n$ and $\text{aff}_{\mathbb{R}^n}(C)$ be the affine hull of C (the intersection of the collection of affine sets containing C). Recall that the relative interior of C in $\mathbb{R}^n$, denoted by $\text{ri}(C)$, is defined by

$$\text{ri}(C) = \{x \in \text{aff}_{\mathbb{R}^n}(C) : \exists \varepsilon > 0, B(x, \varepsilon) \cap \text{aff}_{\mathbb{R}^n}(C) \subset C\},$$

where

$$B(x,\varepsilon) := \left\{y \in \mathbb{R}^n : d(x,y) < \varepsilon\right\}$$

is an open ball around x. Notice that for a convex set C in $\mathbb{R}^n$, we have

$$\mathrm{ri}\,\{\mathrm{conv}_{\mathbb{R}^n}(C)\} \cap \mathbb{R}^n = \mathrm{ri}\,(C)\,.$$

Inspired by this notion, we define the convex-relative interior for sets in $\mathbb{Z}^n$.

Definition 5.1 (*Convex-relative interior in* $\mathbb{Z}^n$) Let set S be a set in $\mathbb{Z}^n$. The convex-relative interior of S in $\mathbb{Z}^n$, denoted by $\mathrm{cri}_{\mathbb{Z}^n}(S)$, is defined by

$$\mathrm{cri}_{\mathbb{Z}^n}(S) = \mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n.$$

A set $S \subset \mathbb{Z}^n$ is relatively open in $\mathbb{Z}^n$, if

$$S = \mathrm{cri}_{\mathbb{Z}^n}(S)$$

and S is regular in $\mathbb{Z}^n$, if $\mathrm{cri}_{\mathbb{Z}^n}(S) \neq \varnothing$.

Note that, for a convex set S in $\mathbb{Z}^n$, it is always true that

$$\mathrm{cri}_{\mathbb{Z}^n}(S) \subset S \subset \mathrm{ccl}_{\mathbb{Z}^n}(S)\,. \tag{5.1}$$

Remark 5.2 For any set $S \subset \mathbb{Z}^n$, the convex-relative interior $\mathrm{cri}_{\mathbb{Z}^n}(S)$ can be alternatively defined by

$$\begin{aligned}\mathrm{cri}_{\mathbb{Z}^n}(S) = \{&x \in \mathrm{aff}_{\mathbb{Z}^n}(S) : \exists\varepsilon > 0 \\ &\text{s.t.}\quad B(x,\varepsilon) \cap \mathrm{aff}_{\mathbb{R}^n}(S) \subset \mathrm{conv}_{\mathbb{R}^n}(S)\}\,.\end{aligned} \tag{5.2}$$

This indicates that the convex-relative interior operator is not a monotone operator, i.e., $S_1 \subseteq S_2$ may not imply that $\mathrm{cri}_{\mathbb{Z}^n}(S_1) \subseteq \mathrm{cri}_{\mathbb{Z}^n}(S_2)$. As an example, consider that $S_1 = \{0\}$ and $S_2 = \{0,1\}$ in $\mathbb{Z}$. Obviously, $S_1 \subseteq S_2$ but $S_1 = \mathrm{cri}_{\mathbb{Z}}(S_1) \supseteq \mathrm{cri}_{\mathbb{Z}}(S_2) = \varnothing$. To guarantee the monotonicity of the convex-relative interior operator, we need to impose the additional condition that $\mathrm{aff}_{\mathbb{Z}^n}(S_1) = \mathrm{aff}_{\mathbb{Z}^n}(S_2)$. Under this condition, from (3.4) and (5.2), we see that

$$S_1 \subseteq S_2 \Rightarrow \mathrm{cri}_{\mathbb{Z}^n}(S_1) \subseteq \mathrm{cri}_{\mathbb{Z}^n}(S_2)\,.$$

Recall the next result in conventional convex analysis.

Lemma 5.3 ([15, Lemma 1.10]) *For any nonempty set* $A \subseteq \mathbb{R}^n$,

$$\mathrm{aff}_{\mathbb{R}^n}(A) = \mathrm{aff}_{\mathbb{R}^n}\left(\overline{A}\right) = \mathrm{aff}_{\mathbb{R}^n}(\mathrm{conv}_{\mathbb{R}^n}(A)) = \mathrm{aff}_{\mathbb{R}^n}\left(\overline{\mathrm{conv}_{\mathbb{R}^n}(A)}\right).$$

A similar result in $\mathbb{Z}^n$ is given below.

Lemma 5.4 *For any nonempty set $S \subseteq \mathbb{Z}^n$*

$$\operatorname{aff}_{\mathbb{Z}^n}(S) = \operatorname{aff}_{\mathbb{Z}^n}(\operatorname{conv}_{\mathbb{Z}^n}(S)).$$

Proof The proof follows from the following relation

$$\begin{aligned}\operatorname{aff}_{\mathbb{Z}^n}(S) &\subseteq \operatorname{aff}_{\mathbb{Z}^n}(\operatorname{conv}_{\mathbb{Z}^n}(S))\\ &= \operatorname{aff}_{\mathbb{R}^n}(\operatorname{conv}_{\mathbb{Z}^n}(S)) \cap \mathbb{Z}^n\\ &= \operatorname{aff}_{\mathbb{R}^n}(\operatorname{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n\\ &= \operatorname{aff}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n\\ &= \operatorname{aff}_{\mathbb{Z}^n}(S).\end{aligned}$$

Theorem 5.5 *For a set $S \subset \mathbb{Z}^n$, $\operatorname{cri}_{\mathbb{Z}^n}(S)$ and $\operatorname{ccl}_{\mathbb{Z}^n}(S)$ are convex sets in $\mathbb{Z}^n$.*

Proof From [14, Theorem 6.2], we know that $\operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S))$ and $\overline{\operatorname{conv}_{\mathbb{R}^n}(S)}$ are convex sets in $\mathbb{R}^n$ having the same affine hull and same dimensionality as $\operatorname{conv}_{\mathbb{R}^n}(S)$. Hence, the sets

$$\operatorname{cri}_{\mathbb{Z}^n}(S) = \operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n$$

and

$$\operatorname{ccl}_{\mathbb{Z}^n}(S) = \overline{\operatorname{conv}_{\mathbb{R}^n}(S)} \cap \mathbb{Z}^n$$

are convex in $\mathbb{Z}^n$.

Remark 5.6 For a nonempty convex set $S \subset \mathbb{Z}^n$, it is possible that $\operatorname{cri}_{\mathbb{Z}^n}(S) = \varnothing$. For such a set, it is possible that $\operatorname{aff}_{\mathbb{Z}^n}(S) \neq \operatorname{aff}_{\mathbb{Z}^n}(\operatorname{cri}_{\mathbb{Z}^n}(S))$. For example, let $S = \{0, 1\} \subset \mathbb{Z}$. Then, $\operatorname{cri}_{\mathbb{Z}}(S) = \operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z} = (0, 1) \cap \mathbb{Z} = \varnothing$ and $\operatorname{ccl}_{\mathbb{Z}^n}(S) = S$. Also notice that $S = \{0, 1\}$ is convex in $\mathbb{Z}$ with

$$\varnothing = \operatorname{aff}_{\mathbb{Z}}(\operatorname{cri}_{\mathbb{Z}}(S)) \subset \operatorname{aff}_{\mathbb{Z}}(S) = \mathbb{Z}.$$

Remark 5.7 Recall from conventional convex analysis [15, Lemma 1.12], for a regular set C in $\mathbb{R}^n$, we have

$$\operatorname{aff}_{\mathbb{R}^n}(\operatorname{ri}(C)) = \operatorname{aff}_{\mathbb{R}^n}(C).$$

However, this may not be true in the discrete case. For example, consider the regular set $S = \{1, 2, 3\}$ in $\mathbb{Z}$. We have

$$\operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S)) = (1, 3),$$

which implies that

$$\operatorname{aff}_{\mathbb{Z}}(\operatorname{cri}_{\mathbb{Z}}(S)) = \{2\} \neq \mathbb{Z} = \operatorname{aff}_{\mathbb{Z}}(S).$$

A necessary and sufficient condition for the desired equality $\mathrm{aff}_{\mathbb{Z}^n}(S) = \mathrm{aff}_{\mathbb{Z}^n}(cri_{\mathbb{Z}^n}(S))$ is given below.

Corollary 5.8 *Let S be a regular set in $\mathbb{Z}^n$. Then,*

$$\mathrm{aff}_{\mathbb{Z}^n}\left(\mathrm{cri}_{\mathbb{Z}^n}(S)\right) = \mathrm{aff}_{\mathbb{R}^n}\left(\mathrm{ri}\left(\mathrm{conv}_{\mathbb{R}^n}(S)\right)\right) \cap \mathbb{Z}^n, \tag{5.3}$$

if and only if

$$\mathrm{aff}_{\mathbb{Z}^n}(S) = \mathrm{aff}_{\mathbb{Z}^n}\left(\mathrm{cri}_{\mathbb{Z}^n}(S)\right).$$

Proof Using (5.3), [14, Theorem 6.2] and [15, Lemma 1.12], we have

$$\begin{aligned}\mathrm{aff}_{\mathbb{Z}^n}\left(\mathrm{cri}_{\mathbb{Z}^n}(S)\right) &\subset \mathrm{aff}_{\mathbb{R}^n}\left(\mathrm{ri}\left(\mathrm{conv}_{\mathbb{R}^n}(S)\right)\right) \cap \mathbb{Z}^n \\ &= \mathrm{aff}_{\mathbb{R}^n}\left(\mathrm{conv}_{\mathbb{R}^n}(S)\right) \cap \mathbb{Z}^n \\ &= \mathrm{aff}_{\mathbb{R}^n}(S) \cap \mathbb{Z}^n \\ &= \mathrm{aff}_{\mathbb{Z}^n}(S).\end{aligned}$$

The proof follows.

Lemma 5.9 *For any set S in $\mathbb{Z}^n$,*

$$\mathrm{aff}_{\mathbb{Z}^n}(S) = \mathrm{aff}_{\mathbb{Z}^n}\left(\mathrm{ccl}_{\mathbb{Z}^n}(S)\right).$$

Proof For $S \subseteq \mathbb{Z}^n$, we have

$$S \subset \overline{\mathrm{conv}_{\mathbb{R}^n}(S)} \subset \mathrm{aff}_{\mathbb{R}^n}(S).$$

Therefore,

$$S \subset \mathrm{ccl}_{\mathbb{Z}^n}(S) \subset \mathrm{aff}_{\mathbb{Z}^n}(S).$$

The desired result follows from Corollary 3.9.

For convenience, we prove the following result on a more general domain Λ^n which can turn into $\mathbb{Z}^n$ or product of $\mathbb{R}$ and $\mathbb{Z}$.

Proposition 5.10 *Let T_i, $i = 1, 2, \cdots, n$, be nonempty closed subsets of reals and Λ^n denote the product $T_1 \times T_2 \times \cdots \times T_n$. For any set S in Λ^n, we have*

$$\overline{\mathrm{conv}_{\mathbb{R}^n}(S)} = \mathrm{conv}_{\mathbb{R}^n}\left(\mathrm{ccl}_{\Lambda^n}(S)\right), \tag{5.4}$$

where

$$\mathrm{ccl}_{\Lambda^n}(S) := \overline{\mathrm{conv}_{\mathbb{R}^n}(S)} \cap \Lambda^n.$$

Proof We may assume that S is closed in $\mathbb{R}^n$ because

$$S \subset \overline{S} \subset \overline{\text{conv}_{\mathbb{R}^n}(S)}$$

implies

$$\overline{\text{conv}_{\mathbb{R}^n}(S)} = \overline{\text{conv}_{\mathbb{R}^n}\left(\overline{S}\right)},$$

and hence, $\text{ccl}_{\Lambda^n}(S) = \text{ccl}_{\Lambda^n}\left(\overline{S}\right)$. For $i = 1, 2, \cdots, n$, let T_i be the ith closed set in the product Λ^n. Then, the complement T_i^C of T_i is either the empty set or the union of countably many open intervals that do not intersect with each other, i.e., $T_i^C = \cup_k I_i^k$, where I_i^k is an open interval, for $k = 1, 2, \cdots$, and $I_i^k \cap I_i^{k'} = \varnothing$ for $k \neq k'$. Define

$$V_i^k := \mathbb{R} \times \mathbb{R} \times \cdots \times I_i^k \times \cdots \times \mathbb{R}.$$

Then, we have

$$\left(\Lambda^n\right)^C = \bigcup_{i,k} V_i^k.$$

Along with $\text{ccl}_{\Lambda^n}(S) = \overline{\text{conv}_{\mathbb{R}^n}(S)} \cap \Lambda^n$, we have

$$\text{ccl}_{\Lambda^n}(S) = \overline{\text{conv}_{\mathbb{R}^n}(S)} \setminus \bigcup_{i,k} V_i^k.$$

Now, we only need to show that

$$\overline{\text{conv}_{\mathbb{R}^n}(S)} \subseteq \text{conv}_{\mathbb{R}^n}\left(\overline{\text{conv}_{\mathbb{R}^n}(S)} \setminus \bigcup_{i,k} V_i^k\right) \tag{5.5}$$

since $\overline{\text{conv}_{\mathbb{R}^n}(S)} \supseteq \text{conv}_{\mathbb{R}^n}(\overline{\text{conv}_{\mathbb{R}^n}(S)} \setminus \cup_{i,k} V_i^k)$ is clear.

Suppose that $x = (x_1, x_2, \cdots, x_n) \in \overline{\text{conv}_{\mathbb{R}^n}(S)}$. There are two cases: (i) $x \notin V_i^k$ for all i, k; (ii) there exists a set $J(x) \subset \{1, 2, \cdots, n\}$ such that, for each $i \in J(x)$, there is a unique k_i satisfying $x \in V_i^{k_i}$. For case (i), $x \in \overline{\text{conv}_{\mathbb{R}^n}(S)} \setminus \cup_{i,k} V_i^k$ and (5.5) follows. For case (ii), we know $J(x) \neq \varnothing$ and for each $i \in J(x)$, the set $I_i^{k_i}$ is a finite interval. Let $I_i^{k_i} := (a_i^{k_i}, b_i^{k_i})$, where $a_i^{k_i}, b_i^{k_i} \in T_i$ for all i, k_i. Without loss of generality, we assume that $1 \in J(x)$. This implies that $x \in V_1^{k_1}$, and $x_1 \in (a_1^{k_1}, b_1^{k_1})$ by definition. Define the sets

$$A_i^{k_i} := \left\{y \in \overline{\text{conv}_{\mathbb{R}^n}(S)} : y_i = a_i^{k_i}\right\}$$

and

$$B_i^{k_i} := \left\{y \in \overline{\text{conv}_{\mathbb{R}^n}(S)} : y_i = b_i^{k_i}\right\}.$$

It is obvious that $A_1^{k_1} \cap V_1^k = \varnothing$ and $B_1^{k_1} \cap V_1^k = \varnothing$ for all $k = 1, 2, \cdots$. We note also that $x \in \overline{\text{conv}_{\mathbb{R}^n}(S)} \cap V_1^{k_1}$ and $S \cap V_1^{k_1} = \varnothing$. Considering the special structure of $V_1^{k_1}$ it is impossible that the set $\overline{\text{conv}_{\mathbb{R}^n}(S)} \cap V_1^{k_1}$ has any extreme points. Actually, $\overline{\text{conv}_{\mathbb{R}^n}(S)} \cap V_1^{k_1}$ is the union of line segments between the points of $A_1^{k_1}$ and $B_1^{k_1}$. Thus, there must exist points $y^{A_1^{k_1}} \in A_1^{k_1}$ and $y^{B_1^{k_1}} \in B_1^{k_1}$ such that x is a convex combination of $y^{A_1^{k_1}}$ and $y^{B_1^{k_1}}$. Note that $1 \notin J\left(y^{A_1^{k_1}}\right)$ and $1 \notin J\left(y^{B_1^{k_1}}\right)$ since $A_1^{k_1} \cap V_1^k = \varnothing$ and $B_1^{k_1} \cap V_1^k = \varnothing$ for all $k = 1, 2, \cdots$. This shows that

$$x \in \text{conv}_{\mathbb{R}^n}\left(\overline{\text{conv}_{\mathbb{R}^n}(S)} \backslash \bigcup_k V_1^k\right).$$

We may assume, w.l.o.g., that $2 \in J\left(y^{A_1^{k_1}}\right)$. This implies that $y^{A_1^{k_1}} \in \overline{\text{conv}_{\mathbb{R}^n}(S)} \cap V_2^{k_2}$. Then, we may conclude the existence of two points $y^{A_1^{k_1}, A_2^{k_2}} \in A_1^{k_1} \cap A_2^{k_2}$ and $y^{A_1^{k_1}, B_2^{k_2}} \in A_1^{k_1} \cap B_2^{k_2}$ such that $y^{A_1^{k_1}}$ is a convex combination of $y^{A_1^{k_1}, A_2^{k_2}}$ and $y^{A_1^{k_1}, B_2^{k_2}}$. Similar arguments can be applied to $y^{B_1^{k_1}}$. This shows that

$$x \in \text{conv}_{\mathbb{R}^n}\left(\overline{\text{conv}_{\mathbb{R}^n}(S)} \backslash \bigcup_k \left(V_1^k \cup V_2^k\right)\right).$$

We can continue this procedure to eliminate all V_i^k from $\overline{\text{conv}_{\mathbb{R}^n}(S)}$. This means we have at most 2^n points in $\overline{\text{conv}_{\mathbb{R}^n}(S)} \backslash \cup_{i,k} V_i^k$ such that x is a convex combination of them. Therefore, $x \in \text{conv}_{\mathbb{R}^n}(\overline{\text{conv}_{\mathbb{R}^n}(S)} \backslash \cup_{i,k} V_i^k)$. This completes the proof.

Proposition 5.11 *If S is an integrally closed set in $\mathbb{Z}^n$, i.e., $S = \text{ccl}_{\mathbb{Z}^n}(S)$, then*

$$\overline{\text{conv}_{\mathbb{R}^n}(S)} = \text{conv}_{\mathbb{R}^n}(S).$$

Corollary 5.12 *For any $S \subset \mathbb{Z}^n$, $\text{ccl}_{\mathbb{Z}^n}(S)$ is an integrally closed set in $\mathbb{Z}^n$, i.e.,*

$$\text{ccl}_{\mathbb{Z}^n}(\text{ccl}_{\mathbb{Z}^n}(S)) = \text{ccl}_{\mathbb{Z}^n}(S). \tag{5.6}$$

Proof For $S \subset \mathbb{Z}^n$, (5.4) implies that

$$\begin{aligned} \text{ccl}_{\mathbb{Z}^n}(\text{ccl}_{\mathbb{Z}^n}(S)) &= \overline{\text{conv}_{\mathbb{R}^n}(\text{ccl}_{\mathbb{Z}^n}(S))} \cap \mathbb{Z}^n \\ &= \overline{\text{conv}_{\mathbb{R}^n}(S)} \cap \mathbb{Z}^n \\ &= \text{ccl}_{\mathbb{Z}^n}(S). \end{aligned}$$

Example 5.13 Consider the convex set $S = \{(0, 1)\} \cup \{(x, 0) : x \in \mathbb{Z}\}$ in $\mathbb{Z}^2$. We know

$$S \neq \overline{\text{conv}_{\mathbb{R}^2}(S)} \cap \mathbb{Z}^2 = \{(x, y) : x \in \mathbb{Z}, \quad y \in \{0, 1\}\}.$$

Clearly, $\overline{\text{conv}_{\mathbb{R}^2}(S)}$ is the closed strip

$$\overline{\text{conv}_{\mathbb{R}^2}(S)} = \{(x, y) : x \in \mathbb{R} \quad \text{and} \quad 0 \leqslant y \leqslant 1\},$$

which does not coincide with the set

$$\text{conv}_{\mathbb{R}^2}(S) = \{(x, y) : x \in \mathbb{R} \quad \text{and} \quad 0 \leqslant y < 1\}.$$

However, we have

$$\overline{\text{conv}_{\mathbb{R}^2}(S)} = \text{conv}_{\mathbb{R}^2}\left(\overline{\text{conv}_{\mathbb{R}^2}(S)} \cap \mathbb{Z}^2\right),$$

i.e., (5.4) holds.

Corollary 5.14 *Let* S *be a nonempty set in* $\mathbb{Z}^n$. *If*

$$\text{ri}\,(\text{conv}_{\mathbb{R}^n}(S)) = \text{conv}_{\mathbb{R}^n}(\text{ri}\,(\text{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n), \tag{5.7}$$

then $\text{cri}_{\mathbb{Z}^n}(S) \neq \varnothing$.

Proof If $\text{cri}_{\mathbb{Z}^n} S = \text{ri}\,(\text{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n = \varnothing$, then the right hand side of (5.7) will be an empty set, while $\text{ri}\,(\text{conv}_{\mathbb{R}^n}(S)) \neq \varnothing$ can be inferred by Hadjisavvas et al. [15, Lemma 1.13]. This completes the proof.

Note that the set $S = (x, x) : x \in \mathbb{Z}$ satisfies (5.7).

Corollary 5.15 *For any set* S *in* $\mathbb{Z}^n$ *satisfying* (5.7),

$$\text{cri}_{\mathbb{Z}^n}(\text{cri}_{\mathbb{Z}^n}(S)) = \text{cri}_{\mathbb{Z}^n}(S).$$

Proof By (5.7) and [15, Relation (1.24), p. 17], we know that

$$\begin{aligned}
\text{cri}_{\mathbb{Z}^n}(\text{cri}_{\mathbb{Z}^n}(S)) &= \text{ri}\left(\text{conv}_{\mathbb{R}^n}\left(\text{ri}\,(\text{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n\right)\right) \cap \mathbb{Z}^n \\
&= \text{ri}\,(\text{ri}\,(\text{conv}_{\mathbb{R}^n}(S))) \cap \mathbb{Z}^n \\
&= \text{ri}\,(\text{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n \\
&= \text{cri}_{\mathbb{Z}^n}(S).
\end{aligned}$$

Remark 5.16 In continuous case, a nonempty convex set always have a nonempty relative interior. However, in the discrete case a nonempty convex set in $\mathbb{Z}^n$ may have an empty convex-relative interior. For example, the nonempty convex set $S = \{0, 1\}$ in $\mathbb{Z}$ has $\text{cri}_{\mathbb{Z}}(S) = \varnothing$. Thus, the convexity of a nonempty set S in $\mathbb{Z}^n$ is not a sufficient condition to have a nonempty convex-relative interior. Furthermore, the relation (5.7) is not a necessary condition for a nonempty convex set S in $\mathbb{Z}^n$ to have nonempty convex-relative interior $\mathrm{cri}_{\mathbb{Z}^n}(S)$. To see this, consider the convex set $S = \{(a,b) : b \in \{0,1\} \text{ and } a \in \mathbb{Z}\} \cup \{(0,2)\}$ in $\mathbb{Z}^2$ with

$$\begin{aligned}\mathrm{conv}_{\mathbb{R}^n}\left(\mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n\right) &= \{(a,b) : b = 1 \quad \text{and} \quad a \in \mathbb{R}\} \\ &\neq \mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S)) \\ &= \{(a,b) : b \in (0,2) \quad \text{and} \quad a \in \mathbb{R}\}.\end{aligned}$$

Theorem 5.17 *For any set S in* $\mathbb{Z}^n$,

$$\mathrm{cri}_{\mathbb{Z}^n}(\mathrm{ccl}_{\mathbb{Z}^n}(S)) = \mathrm{cri}_{\mathbb{Z}^n}(S). \tag{5.8}$$

and

$$\mathrm{ccl}_{\mathbb{Z}^n}(\mathrm{cri}_{\mathbb{Z}^n}(S)) \subseteq \mathrm{ccl}_{\mathbb{Z}^n}(S).$$

Moreover, if S satisfies (5.7)*, then*

$$\mathrm{ccl}_{\mathbb{Z}^n}(\mathrm{cri}_{\mathbb{Z}^n}(S)) = \mathrm{ccl}_{\mathbb{Z}^n}(S). \tag{5.9}$$

Proof By Rockafellar [14, Theorem 6.3] and (5.4), we have

$$\begin{aligned}\mathrm{cri}_{\mathbb{Z}^n}(\mathrm{ccl}_{\mathbb{Z}^n}(S)) &= \mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(\mathrm{ccl}_{\mathbb{Z}^n}(S))) \cap \mathbb{Z}^n \\ &= \mathrm{ri}\left(\overline{\mathrm{conv}_{\mathbb{R}^n}(S)}\right) \cap \mathbb{Z}^n \\ &= \mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n \\ &= \mathrm{cri}_{\mathbb{Z}^n}(S).\end{aligned}$$

Moreover, for $S \subset \mathbb{Z}^n$, we have

$$\begin{aligned}\mathrm{ccl}_{\mathbb{Z}^n}(\mathrm{cri}_{\mathbb{Z}^n}(S)) &= \overline{\mathrm{conv}_{\mathbb{R}^n}(\mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S)) \cap \mathbb{Z}^n)} \cap \mathbb{Z}^n \\ &\subseteq \overline{\mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S))} \cap \mathbb{Z}^n \\ &= \overline{\mathrm{conv}_{\mathbb{R}^n}(S)} \cap \mathbb{Z}^n \\ &= \mathrm{ccl}_{\mathbb{Z}^n}(S).\end{aligned}$$

If (5.7) holds, the last inclusion relation above turns into equality relation. This proves (5.9).

Theorem 5.18 *Let* $S_1 \subset \mathbb{Z}^n$ *and* $S_2 \subset \mathbb{Z}^m$ *be convex sets, then*

$$\mathrm{cri}_{\mathbb{Z}^{n+m}}(S_1 \times S_2) = \mathrm{cri}_{\mathbb{Z}^n}(S_1) \times \mathrm{cri}_{\mathbb{Z}^m}(S_2).$$

Proof By Hadjisavvas et al. [15, Eq. (1.22)], we have

$$\mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S_1) \times \mathrm{conv}_{\mathbb{R}^n}(S_2)) = \mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S_1)) \times \mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(S_2)).$$

Consequently,

$$\begin{aligned}\operatorname{cri}_{\mathbb{Z}^{n+m}}(S_1 \times S_2) &= \operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S_1 \times S_2)) \cap \mathbb{Z}^{n+m} \\ &= \operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S_1) \times \operatorname{conv}_{\mathbb{R}^n}(S_2)) \cap \mathbb{Z}^{n+m} \\ &= (\operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S_1)) \times \operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S_2))) \cap \mathbb{Z}^{n+m} \\ &= \left(\operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S_1)) \cap \mathbb{Z}^n\right) \times \left(\operatorname{ri}(\operatorname{conv}_{\mathbb{R}^n}(S_2)) \cap \mathbb{Z}^m\right) \\ &= \operatorname{cri}_{\mathbb{Z}^n}(S_1) \times \operatorname{cri}_{\mathbb{Z}^m}(S_2).\end{aligned}$$

6 Constructing a Convex Function on $\mathbb{R}^n$ Using a Convex Function on $\mathbb{Z}^n$

We know by Lemma 2.7 that, for any convex set S in $\mathbb{Z}^n$, there is a convex set C in $\mathbb{R}^n$ such that $S = C \cap \mathbb{Z}^n$. In this section, we show that, for any convex function f defined on $\mathbb{Z}^n$, there is a convex function defined on $\mathbb{R}^n$ whose restriction to $\mathbb{Z}^n$ coincides with f.

Let f be a function on the discrete domain $\mathbb{Z}^n$. By [14, Theorem 5.3], we know that the function given by

$$\operatorname{conv}_{\mathbb{R}^n}(f)(x) := \inf\left\{y \in \mathbb{R} : (x, y) \in \operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))\right\} \tag{6.1}$$

is a convex function on $\mathbb{R}^n$ with the understanding that the $\inf \varnothing = +\infty$. Hereafter, $\operatorname{conv}_{\mathbb{R}^n}(f)$ is called the convex hull function of f *on* $\mathbb{R}^n$.

Observe that, for the inner epigraph of $\operatorname{conv}_{\mathbb{R}^n}(f)$,

$$\operatorname{epi}(\widetilde{\operatorname{conv}_{\mathbb{R}^n}}(f)) \subseteq \operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f)) \subseteq \operatorname{epi}(\operatorname{conv}_{\mathbb{R}^n}(f)). \tag{6.2}$$

Lemma 6.1 *For any function $f : \mathbb{Z}^n \to [-\infty, \infty]$,*

$$\operatorname{dom}(\operatorname{conv}_{\mathbb{R}^n}(f)) = \operatorname{conv}_{\mathbb{R}^n}(\operatorname{dom}(f)). \tag{6.3}$$

Proof Let $x \in \operatorname{dom}(\operatorname{conv}_{\mathbb{R}^n}(f))$, then $(x, r) \in \operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))$ for any $r > \operatorname{conv}_{\mathbb{R}^n}(f)(x)$. This implies that

$$\begin{aligned}x &\in \pi_1\left(\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))\right) \\ &= \operatorname{conv}_{\mathbb{R}^n}(\pi_1(\operatorname{epi}(f))) \\ &= \operatorname{conv}_{\mathbb{R}^n}(\operatorname{dom}(f)).\end{aligned}$$

Conversely, if $x \in \operatorname{conv}_{\mathbb{R}^n}(\operatorname{dom}(f))$, then we have

$$\begin{aligned}x &\in \operatorname{conv}_{\mathbb{R}^n}(\pi_1(\operatorname{epi}(f))) \\ &= \pi_1\left(\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))\right) \\ &\subseteq \pi_1(\operatorname{epi}(\operatorname{conv}_{\mathbb{R}^n}(f))) \\ &= \operatorname{dom}(\operatorname{conv}_{\mathbb{R}^n}(f)).\end{aligned}$$

Definition 6.2 (*Convex hull function on* $\mathbb{Z}^n$) For any function $f : \mathbb{Z}^n \to [-\infty, \infty]$, the function $f_c : \mathbb{Z}^n \to [-\infty, \infty]$ defined by

$$f_c(x) = \text{conv}_{\mathbb{R}^n}(f)|_{\mathbb{Z}^n}(x) \tag{6.4}$$

is called the convex hull function of f on $\mathbb{Z}^n$, where $\text{conv}_{\mathbb{R}^n}(f)|_{\mathbb{Z}^n}$ denotes the restriction of the function $\text{conv}_{\mathbb{R}^n}(f)$ to $\mathbb{Z}^n$.

Notice that

$$f_c(x) = \inf\{y \in \mathbb{R} : (x, y) \in \text{conv}_{\mathbb{Z}^n \times \mathbb{R}}(\text{epi}(f))\}, \tag{6.5}$$

where

$$\text{conv}_{\mathbb{Z}^n \times \mathbb{R}}(\text{epi}(f)) := \text{conv}_{\mathbb{R}^{n+1}}(\text{epi}(f)) \cap \left(\mathbb{Z}^n \times \mathbb{R}\right).$$

Lemma 6.3 *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$, f_c *is the greatest convex function on* $\mathbb{Z}^n$ *dominated by* f. *Moreover, it follows that*

$$\widetilde{\text{epi}(f_c)} \subseteq \text{conv}_{\mathbb{Z}^n \times \mathbb{R}}(\text{epi}(f)) \subseteq \text{epi}(f_c), \tag{6.6}$$

$$\text{dom}(f_c) = \text{dom}(\text{conv}_{\mathbb{R}^n}(f)) \cap \mathbb{Z}^n = \text{conv}_{\mathbb{Z}^n}(\text{dom}(f)), \tag{6.7}$$

and

$$\text{cri}_{\mathbb{Z}^n}(\text{dom}(f_c)) = \text{cri}_{\mathbb{Z}^n}(\text{dom}(f)). \tag{6.8}$$

Proof If h is convex on $\mathbb{Z}^n$ and $h \leqslant f$, then

$$\text{conv}_{\mathbb{Z}^n \times \mathbb{R}}(\text{epi}(f)) \subseteq \text{conv}_{\mathbb{Z}^n \times \mathbb{R}}(\text{epi}(h)) = \text{epi}(h).$$

This shows that f_c is the greatest convex function on $\mathbb{Z}^n$ dominated by f. Using (6.5), we can easily verify (6.6). To show (6.7), (6.3) and (6.4) lead to

$$\begin{aligned}\text{dom}(f_c) &= \text{dom}(\text{conv}_{\mathbb{R}^n}(f)) \cap \mathbb{Z}^n \\ &= \text{conv}_{\mathbb{R}^n}(\text{dom}(f)) \cap \mathbb{Z}^n \\ &= \text{conv}_{\mathbb{Z}^n}(\text{dom}(f)).\end{aligned}$$

Then, (6.8) follows by

$$\begin{aligned}\text{cri}_{\mathbb{Z}^n}(\text{dom}(f_c)) &= \text{ri}(\text{conv}_{\mathbb{R}^n}(\text{dom} f_c)) \cap \mathbb{Z}^n \\ &= \text{ri}\left(\text{conv}_{\mathbb{R}^n}\left(\text{conv}_{\mathbb{R}^n}(\text{dom}(f)) \cap \mathbb{Z}^n\right)\right) \cap \mathbb{Z}^n \\ &\subseteq \text{ri}(\text{conv}_{\mathbb{R}^n}(\text{conv}_{\mathbb{R}^n}(\text{dom}(f)))) \cap \mathbb{Z}^n \\ &= \text{ri}(\text{conv}_{\mathbb{R}^n}(\text{dom}(f))) \cap \mathbb{Z}^n \\ &= \text{cri}_{\mathbb{Z}^n}(\text{dom}(f)).\end{aligned}$$

On the other hand, (6.7) and Lemma 5.4 imply that

$$\operatorname{dom}(f) \subseteq \operatorname{dom}(f_c)$$

and

$$\begin{aligned}\operatorname{aff}_{\mathbb{Z}^n}(\operatorname{dom}(f)) &= \operatorname{aff}_{\mathbb{Z}^n}(\operatorname{conv}_{\mathbb{Z}^n}(\operatorname{dom}(f))) \\ &= \operatorname{aff}_{\mathbb{Z}^n}(\operatorname{dom}(f_c)).\end{aligned}$$

Using the monotonicity of the operator $\operatorname{cri}_{\mathbb{Z}^n}$, we have

$$\operatorname{cri}_{\mathbb{Z}^n}(\operatorname{dom}(f)) \subseteq \operatorname{cri}_{\mathbb{Z}^n}(\operatorname{dom} f_c).$$

The proof then follows.

Remark 6.4 This lemma and Corollary 4.23 offer an alternative representation of the function f_c by

$$f_c(x) = \sup\left\{h(x) : h \leqslant f \quad \text{and} \quad h \text{ is a convex function on } \mathbb{Z}^n\right\}. \tag{6.9}$$

Observe that

$$\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f)) \subseteq \operatorname{epi}(\operatorname{conv}_{\mathbb{R}^n}(f)) \subseteq \overline{\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))}. \tag{6.10}$$

Together with [14, Theorem 6.3], we have

$$\operatorname{ri}\left(\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))\right) = \operatorname{ri}(\operatorname{epi}(\operatorname{conv}_{\mathbb{R}^n}(f))) \tag{6.11}$$

and

$$\operatorname{conv}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f)) \subseteq \operatorname{epi}(f_c) \subseteq \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f)), \tag{6.12}$$

where

$$\operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f)) := \overline{\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))} \cap \left(\mathbb{Z}^n \times \mathbb{R}\right).$$

Theorem 6.5 *f is a convex function on $\mathbb{Z}^n$ if and only if*

$$f_c = f.$$

Proof Corollary 4.8 proves the necessity part. For the sufficiency part, if f is convex on $\mathbb{Z}^n$, then

$$\operatorname{epi}(f) = \operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f)) \cap \left(\mathbb{Z}^n \times \mathbb{R}\right).$$

Therefore, for any $x \in \mathbb{Z}^n$, $\{y \in \mathbb{R} : (x, y) \in \operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))\} = \{y \in \mathbb{R} : (x, y) \in \operatorname{epi}(f)\}$. The rest of the proof follows from (4.2) and (6.1).

Corollary 6.6 *For a convex function f on $\mathbb{Z}^n$*

$$\text{dom}\,(f_c) = \text{dom}\,(f) \quad \text{and} \quad \text{epi}\,(f_c) = \text{epi}\,(f)\,.$$

Lemma 6.7 *Let $f : \mathbb{Z}^n \to [-\infty, \infty]$ be a function with nonempty effective domain. If* $\text{conv}_{\mathbb{R}^n}(f)(x_0) = -\infty$ *for some* $x_0 \in \text{dom}\,(\text{conv}_{\mathbb{R}^n}(f))$*, then* $\text{conv}_{\mathbb{R}^n}(f)(x) = -\infty$ *for all* $x \in ri\,(\text{dom}\,(\text{conv}_{\mathbb{R}^n}(f)))$.

Proof Since $\text{conv}_{\mathbb{R}^n}(f)(x_0) = -\infty$ at $x_0 \in \text{dom}\,(\text{conv}_{\mathbb{R}^n}(f))$, by Rockafellar [14, Theorem 6.4], we know that, for any $x \in \text{ri}\,(\text{dom}\,(\text{conv}_{\mathbb{R}^n}(f)))$, there is a scalar $\eta > 1$ such that

$$y = (1-\eta)x_0 + \eta x \in \text{dom}\,(\text{conv}_{\mathbb{R}^n}(f))\,.$$

Letting $\eta^{-1} = \lambda$, we have

$$x = \lambda x_0 + (1-\lambda)y, \quad \lambda \in (0, 1)\,.$$

The convexity of $\text{conv}_{\mathbb{R}^n}(f)$ and Theorem 4.18 lead to

$$\text{conv}_{\mathbb{R}^n}(f)(x) < \lambda\alpha + (1-\lambda)\beta,$$

for all $\alpha \in \mathbb{R}$ and $\beta > \text{conv}_{\mathbb{R}^n}(f)(y)$. This is possible only if $\text{conv}_{\mathbb{R}^n}(f)(x) = -\infty$. The proof follows.

Theorem 6.8 *If f is a proper convex function on $\mathbb{Z}^n$, then $\text{conv}_{\mathbb{R}^n}(f)$ is a proper convex function on $\mathbb{R}^n$.*

Proof Since f is a proper convex function on $\mathbb{Z}^n$, we have $f > -\infty$ and, by Theorem 6.5,

$$f(x) = \text{conv}_{\mathbb{R}^n}(f)(x) > -\infty \quad \text{for every} \quad x \in \text{dom}\,(f)\,.$$

This is especially true for all $x \in \text{dom}\,(f) \cap \text{ri}\,(\text{dom}\,(\text{conv}_{\mathbb{R}^n}(f)))$. By the contrapositive of Lemma 6.7, we know

$$\text{conv}_{\mathbb{R}^n}(f)(x) > -\infty \quad \text{for every} \quad x \in \text{dom}\,(\text{conv}_{\mathbb{R}^n}(f))\,.$$

Moreover, by the properness of f on $\mathbb{Z}^n$, we have at least one $x_1 \in \mathbb{Z}^n$ such that $f(x_1) = \text{conv}_{\mathbb{R}^n}(f)(x_1) < \infty$. Thus, $\text{conv}_{\mathbb{R}^n}(f)$ is a proper convex function on $\mathbb{R}^n$.

The next result is a direct consequence of Corollary 4.8 and (6.4).

Corollary 6.9 *For any function $f : \mathbb{Z}^n \to [-\infty, \infty]$, if $\text{conv}_{\mathbb{R}^n}(f)$ is a proper convex function on $\mathbb{R}^n$, then f_c is a proper convex function on $\mathbb{Z}^n$.*

Using the contrapositive of the above given result, we get the following:

Corollary 6.10 *If f is an improper convex function on $\mathbb{Z}^n$, then* $\mathrm{conv}_{\mathbb{R}^n}(f)$ *is an improper convex function on* $\mathbb{R}^n$.

Recall the next result in conventional convex analysis.

Lemma 6.11 ([15, Lemma 1.35]) *If $g : \mathbb{R}^n \to [-\infty, \infty]$ is a convex function with* $\mathrm{dom}\,(g) \neq \varnothing$, *then* $\mathrm{ri}\,(\mathrm{epi}\,(g)) \neq \varnothing$ *and*

$$\mathrm{ri}\,(\mathrm{epi}\,(g)) = \Big\{(x, y) \in \mathbb{R}^{n+1} : g(x) < y \quad \text{and} \quad x \in \mathrm{ri}\,(\mathrm{dom}\,(g))\Big\}.$$

This leads to the next result.

Lemma 6.12 *If $f : \mathbb{Z}^n \to [-\infty, \infty]$ is convex on $\mathbb{Z}^n$ with* $\mathrm{cri}_{\mathbb{Z}^n}(\mathrm{dom}\,(f)) \neq \varnothing$, *then* $\mathrm{cri}_{\mathbb{Z}^n \times \mathbb{R}}(\mathrm{epi}\,(f)) \neq \varnothing$ *and*

$$\mathrm{cri}_{\mathbb{Z}^n \times \mathbb{R}}(\mathrm{epi}\,(f)) = \Big\{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : f(x) < y \quad \text{and} \quad x \in \mathrm{cri}_{\mathbb{Z}^n}(\mathrm{dom}\,(f))\Big\}.$$

Proof Since f is convex on $\mathbb{Z}^n$, $f(x) = \mathrm{conv}_{\mathbb{R}^n}(f)(x)$ for all $x \in \mathbb{Z}^n$. If

$$(x, y) \in \Big\{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : f(x) < y \quad \text{and} \quad x \in \mathrm{cri}_{\mathbb{Z}^n}(\mathrm{dom}\,(f))\Big\},$$

then

$$\begin{aligned} x &\in \mathrm{cri}_{\mathbb{Z}^n}(\mathrm{dom}\,(f)) \\ &= \mathrm{ri}\,(\mathrm{conv}_{\mathbb{R}^n}(\mathrm{dom}\,(f))) \cap \mathbb{Z}^n \\ &= \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{conv}_{\mathbb{R}^n}(f))) \cap \mathbb{Z}^n. \end{aligned}$$

Since $f(x) = \mathrm{conv}_{\mathbb{R}^n}(f)(x) < y$, the preceding lemma along with (6.11) implies

$$\begin{aligned} (x, y) &\in \mathrm{ri}\,(\mathrm{epi}\,(\mathrm{conv}_{\mathbb{R}^n}(f))) \cap (\mathbb{Z}^n \times \mathbb{R}) \\ &= \mathrm{ri}\,\big(\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}\,(f))\big) \cap \big(\mathbb{Z}^n \times \mathbb{R}\big) \\ &= \mathrm{cri}_{\mathbb{Z}^n \times \mathbb{R}}(\mathrm{epi}\,(f)). \end{aligned}$$

Using the same arguments, we can verify that

$$\mathrm{cri}_{\mathbb{Z}^n \times \mathbb{R}}(\mathrm{epi}\,(f)) \subseteq \Big\{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : f(x) < y \quad \text{and} \quad x \in \mathrm{cri}_{\mathbb{Z}^n}(\mathrm{dom}\,(f))\Big\}.$$

Definition 6.13 (*Positively homogeneous function in $\mathbb{Z}^n$*) A function f on $\mathbb{Z}^n$ is said to be positively homogeneous (of degree 1) if

$$f(\lambda x) = \lambda f(x)$$

holds for all $x \in \mathbb{Z}^n$ and $\lambda > 0$ such that $\lambda x \in \mathbb{Z}^n$.

Theorem 6.14 *A function f on $\mathbb{Z}^n$ is positively homogeneous if and only if* $\mathrm{epi}\,(f)$ *is a cone in $\mathbb{Z}^n \times \mathbb{R}$.*

7 Convex-Lower Semi Continuity on $\mathbb{Z}^n$

In this section, we define the concept of convex-lower semi continuity for functions defined on $\mathbb{Z}^n$. This concept will play an essential role in studying the dual representation of functions on $\mathbb{Z}^n$.

Recall the definition of lower-semi-continuity in classical convex analysis.

Definition 7.1 (*l.s.c. Hull function on* $\mathbb{R}^n$) For a function $g : \mathbb{R}^n \to [-\infty, \infty]$, the function $\overline{g} : \mathbb{R}^n \to [-\infty, \infty]$ defined by

$$\overline{g}(x) := \inf \left\{ y \in \mathbb{R} : (x, y) \in \overline{\text{epi}\,(g)} \right\} \tag{7.1}$$

is called the lower-semi-continuous hull (or l.s.c. hull) function of g on $\mathbb{R}^n$. We say $g : \mathbb{R}^n \to [-\infty, \infty]$ is lower-semi-continuous, if $\overline{g}(x) = g(x)$ for all $x \in \mathbb{R}^n$.

It is clear that, for a function $g : \mathbb{R}^n \to [-\infty, \infty]$, epi $(\overline{g}) = \overline{\text{epi}\,(g)}$ and dom $(g) \subseteq$ dom $(\overline{g}) \subseteq \overline{\text{dom}\,(g)}$, see [15, Lemma 1.31]. Also notice that, for a function $f : \mathbb{Z}^n \to [-\infty, \infty]$, (5.4) leads to that

$$\overline{\text{conv}_{\mathbb{R}^{n+1}}\,(\text{epi}\,(f))} = \text{conv}_{\mathbb{R}^{n+1}} \left(\overline{\text{conv}_{\mathbb{R}^{n+1}}\,(\text{epi}\,(f))} \cap \left(\mathbb{Z}^n \times \mathbb{R} \right) \right). \tag{7.2}$$

Definition 7.2 (*l.s.c.-Convex hull function on* $\mathbb{Z}^n$) For a function $f : \mathbb{Z}^n \to [-\infty, \infty]$, the function $f_{\text{lsc}} : \mathbb{R}^n \to [-\infty, \infty]$ defined by

$$f_{\text{lsc}}(x) := \inf \left\{ y \in \mathbb{R} : (x, y) \in \overline{\text{conv}_{\mathbb{R}^{n+1}}(\text{epi}\,(f))} \right\} \tag{7.3}$$

is called the l.s.c.-convex hull function of f on $\mathbb{R}^n$. The l.s.c.-convex hull function of f on $\mathbb{Z}^n$, denoted $f_{\overline{c}}$, is defined to be the restriction of f_{lsc} to $\mathbb{Z}^n$, i.e.,

$$f_{\text{lsc}}|_{\mathbb{Z}^n} = f_{\overline{c}}, \tag{7.4}$$

where $f_{\text{lsc}}|_{\mathbb{Z}^n}$ is the restriction of f_{lsc} to $\mathbb{Z}^n$. We say the function $f : \mathbb{Z}^n \to [-\infty, \infty]$ is convex-lower semi continuous or convex-l.s.c., if $f_{\overline{c}}(x) = f(x)$ for all $x \in \mathbb{Z}^n$. In particular, we say $f : \mathbb{Z}^n \to [-\infty, \infty]$ is convex-l.s.c. at a point $x_0 \in \mathbb{Z}^n$ if $f_{\overline{c}}(x_0) = f(x_0)$ holds.

From (7.3), we see that the function $f_{\overline{c}}$ can alternatively be expressed as

$$f_{\overline{c}}(x) = \inf \{ y \in \mathbb{R} : (x, y) \in \text{ccl}_{\mathbb{Z}^n \times \mathbb{R}}\,(\text{epi}\,(f)) \}, \quad \forall x \in \mathbb{Z}^n. \tag{7.5}$$

Lemma 7.3 *For any function $f : \mathbb{Z}^n \to [-\infty, \infty]$, the l.s.c.-convex hull function $f_{\overline{c}}$ is the greatest convex-l.s.c. function on $\mathbb{Z}^n$ dominated by f. Moreover,*

$$\text{epi}\,(f_{\overline{c}}) = \text{ccl}_{\mathbb{Z}^n \times \mathbb{R}}\,(\text{epi}\,(f_c)) = \text{ccl}_{\mathbb{Z}^n \times \mathbb{R}}\,(\text{epi}\,(f)), \tag{7.6}$$

$$\text{dom}\,(f_c) \subseteq \text{dom}\,(f_{\overline{c}}) \subseteq \text{ccl}_{\mathbb{Z}^n}\,(\text{dom}\,(f)), \tag{7.7}$$

and

$$\operatorname{cri}_{\mathbb{Z}^n}(\operatorname{dom}(f_c)) = \operatorname{cri}_{\mathbb{Z}^n}(\operatorname{dom}(f)) = \operatorname{cri}_{\mathbb{Z}^n}(\operatorname{dom}(f_{\bar{c}})) .$$

Proof From (6.10), we have

$$\operatorname{epi}(f_{\mathrm{lsc}}) = \overline{\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))} = \overline{\operatorname{epi}(\operatorname{conv}_{\mathbb{R}^n}(f))}. \tag{7.8}$$

From (7.2), we further have

$$\begin{aligned}\operatorname{epi}(f_{\bar{c}}) &= \overline{\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi}(f))} \cap \left(\mathbb{Z}^n \times \mathbb{R}\right) = \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f))\\ &= \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f_c)) .\end{aligned} \tag{7.9}$$

Denote by $\gamma = f_{\bar{c}}$ and apply (7.2), we see that

$$\begin{aligned}\operatorname{epi}(\gamma_{\bar{c}}) &= \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(\gamma))\\ &= \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f_{\bar{c}}))\\ &= \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f)))\\ &= \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f))\\ &= \operatorname{epi}(f_{\bar{c}})\\ &= \operatorname{epi}(\gamma) .\end{aligned}$$

Hence the function $f_{\bar{c}}$ is convex-l.s.c. on $\mathbb{Z}^n$. If $h \leqslant f$ and h is convex-l.s.c. on $\mathbb{Z}^n$, then

$$\operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(f)) \subseteq \operatorname{ccl}_{\mathbb{Z}^n\times\mathbb{R}}(\operatorname{epi}(h)) = \operatorname{epi}(h_{\bar{c}}) = \operatorname{epi}(h) .$$

It follows that $h \leqslant f_{\bar{c}}$.

Since $\operatorname{epi}(f_c) \subseteq \operatorname{epi}(f_{\bar{c}})$, we know that

$$\operatorname{dom}(f_c) = \pi_1(\operatorname{epi}(f_c)) \subseteq \pi_1(\operatorname{epi}(f_{\bar{c}})) = \operatorname{dom}(f_{\bar{c}}) .$$

The relation of $\operatorname{dom}(f_{\bar{c}}) \subseteq \operatorname{ccl}_{\mathbb{Z}^n}(\operatorname{dom}(f))$ can be obtained from (6.3) and

$$\begin{aligned}\operatorname{dom}(f_{\bar{c}}) &= \pi_1(\operatorname{epi}(f_{\mathrm{lsc}})) \cap \mathbb{Z}^n\\ &= \pi_1\left(\overline{\operatorname{epi}(\operatorname{conv}_{\mathbb{R}^n}(f))}\right) \cap \mathbb{Z}^n\\ &\subseteq \overline{\pi_1(\operatorname{epi}(\operatorname{conv}_{\mathbb{R}^n}(f)))} \cap \mathbb{Z}^n\\ &= \overline{\operatorname{dom}(\operatorname{conv}_{\mathbb{R}^n}(f))} \cap \mathbb{Z}^n\\ &= \overline{\operatorname{conv}_{\mathbb{R}^n}(\operatorname{dom}(f))} \cap \mathbb{Z}^n\\ &= \operatorname{ccl}_{\mathbb{Z}^n}(\operatorname{dom}(f)) .\end{aligned}$$

Finally, Lemma 5.9 and (7.7) lead to that

$$\text{aff}_{\mathbb{Z}^n}(\text{dom}(f)) = \text{aff}_{\mathbb{Z}^n}(\text{dom}(f_c)) = \text{aff}_{\mathbb{Z}^n}(\text{dom}(f_{\overline{c}})).$$

Using Remark (5.2), (7.7), and Theorem 5.17, we have

$$\begin{aligned}\text{cri}_{\mathbb{Z}^n}(\text{dom}(f)) &\subseteq \text{cri}_{\mathbb{Z}^n}(\text{dom}(f_c))\\ &\subseteq \text{cri}_{\mathbb{Z}^n}(\text{dom}(f_{\overline{c}}))\\ &\subseteq \text{cri}_{\mathbb{Z}^n}(\text{ccl}_{\mathbb{Z}^n}(\text{dom}(f)))\\ &\subseteq \text{cri}_{\mathbb{Z}^n}(\text{dom}(f)).\end{aligned}$$

This completes the proof.

For a function $f : \mathbb{Z}^n \to [-\infty, \infty]$, the previous lemma and Corollary 4.23 lead to

$$f_{\overline{c}}(x) = \sup\left\{h(x) : h \leqslant f \quad \text{and} \quad h \text{ is convex-l.s.c. on } \mathbb{Z}^n\right\}, \quad \forall x \in \mathbb{Z}^n. \quad (7.10)$$

Combining (7.1) and (7.8), we have

$$f_{\text{lsc}} = \overline{\text{conv}_{\mathbb{R}^n}(f)}, \quad (7.11)$$

and, therefore,

$$f_{\text{lsc}}(x) = \sup\left\{h(x) : h \leqslant \text{conv}_{\mathbb{R}^n}(f) \quad \text{and} \quad h \text{ is convex and l.s.c. on } \mathbb{R}^n\right\}, \quad \forall x \in \mathbb{R}^n.$$

In preparation for the next result, we recall the following definition from continuous case:

Definition 7.4 ([15, Definition 1.14] and [15, Definition 1.21]) Let $g : \mathbb{R}^n \to [-\infty, \infty]$ be a function. The convex hull g_c and the l.s.c.-convex hull $g_{\overline{c}}$ of the function g are defined by

$$g_c(x) := \inf\{r : (x, r) \in \text{conv}_{\mathbb{R}^n}(\text{epi } f)\}$$

and

$$g_{\overline{c}}(x) := \inf\left\{r : (x, r) \in \overline{\text{conv}_{\mathbb{R}^n}(\text{epi } f)}\right\},$$

respectively.

It is known from [15, Lemma 1.26] and [15, Lemma 1.37] that g_c is the greatest convex function on $\mathbb{R}^n$ dominated by g and $g_{\overline{c}}$ is the greatest convex and lower-semi-continuous function dominated by g.

Lemma 7.5 *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$, *we have*

$$\operatorname{conv}_{\mathbb{R}^n}(f_{\overline{c}}) = \overline{\operatorname{conv}_{\mathbb{R}^n}(f)} = (\operatorname{conv}_{\mathbb{R}^n}(f))_{\overline{c}}\,. \tag{7.12}$$

Proof By the convexity of $\operatorname{conv}_{\mathbb{R}^n}(f)$, we know that

$$\operatorname{conv}_{\mathbb{R}^n}(f) = (\operatorname{conv}_{\mathbb{R}^n}(f))_c\,. \tag{7.13}$$

By (7.2), (7.1), and (7.8), we have

$$\operatorname{conv}_{\mathbb{R}^n}(f_{\overline{c}}) = \overline{\operatorname{conv}_{\mathbb{R}^n}(f)}.$$

From [15, Relation 1.66, p. 41], we know that, for any function $g : \mathbb{R}^n \to [-\infty, \infty]$,

$$g_{\overline{c}} = \overline{g_c}.$$

This and (7.13) imply that

$$\overline{\operatorname{conv}_{\mathbb{R}^n}(f)} = (\operatorname{conv}_{\mathbb{R}^n}(f))_{\overline{c}}\,.$$

Theorem 7.6 *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$, f *is convex-l.s.c. on* $\mathbb{Z}^n$ *if and only* $\operatorname{epi}(f)$ *is integrally closed in* $\mathbb{Z}^n \times \mathbb{R}$, *i.e.,* $\operatorname{ccl}_{\mathbb{Z}^n \times \mathbb{R}}(\operatorname{epi}(f)) = \operatorname{epi}(f)$.

Proof It is a direct consequence of (7.5) and Lemma 7.3.

Corollary 7.7 *For a function* $f : \mathbb{Z}^n \to [-\infty, \infty]$, f *is convex-l.s.c. on* $\mathbb{Z}^n$, *then the level set*

$$L(f, \alpha) = \left\{x \in \mathbb{Z}^n : f(x) \leqslant \alpha\right\}$$

is integrally closed in $\mathbb{Z}^n$ *for each* $\alpha \in \mathbb{R}$, *i.e.,*

$$\operatorname{ccl}_{\mathbb{Z}^n}(L(f, \alpha)) = L(f, \alpha)\,, \quad \forall \alpha \in \mathbb{R}.$$

Proof If f is convex-l.s.c. on $\mathbb{Z}^n$, then the previous theorem says that

$$\operatorname{ccl}_{\mathbb{Z}^n \times \mathbb{R}}(\operatorname{epi}(f)) = \operatorname{epi}(f)\,.$$

Together with (7.2), we have

$$\overline{\operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi} f)} = \operatorname{conv}_{\mathbb{R}^{n+1}}(\operatorname{epi} f)\,.$$

Consequently,

$$\overline{\operatorname{conv}_{\mathbb{R}^n}(f)} = \operatorname{conv}_{\mathbb{R}^n}(f)\,,$$

which means $\text{conv}_{\mathbb{R}^n}(f)$ is l.s.c. on $\mathbb{R}^n$. Now, for any $\alpha \in \mathbb{R}$ and $x \in \text{ccl}_{\mathbb{Z}^n}(L(f, \alpha))$, we know $x \in \mathbb{Z}^n$ and there is a sequence of points $\{x_n\}$ in $\text{conv}_{\mathbb{R}^n}(L(f, \alpha))$ converging to x. Since

$$\text{conv}_{\mathbb{R}^n}(L(f, \alpha)) \subseteq L(\text{conv}_{\mathbb{R}^n}(f), \alpha),$$

we have

$$\text{conv}_{\mathbb{R}^n}(f)(x_n) \leqslant \alpha.$$

The lower semi-continuity of $\text{conv}_{\mathbb{R}^n}(f)$ at x_n implies that,

$$f(x) = \text{conv}_{\mathbb{R}^n}(f)(x) \leqslant \alpha.$$

Therefore, $x \in L(f, \alpha)$.

Remark 7.8 We know by Hadjisavvas et al. [15, Theorem 1.7] that $g : \mathbb{R}^n \to [-\infty, \infty]$ is l.s.c. if and only if epi (g) is closed. However, for a function $f : \mathbb{Z}^n \to [-\infty, \infty]$, the closeness of epi (f) may not imply that f is convex-l.s.c. Consider the function $f : \mathbb{Z}^2 \to [-\infty, \infty]$ defined by

$$f(x, y) = \begin{cases} 0, & \text{if } x = y = 0, \\ \infty, & \text{if } x = 0 \quad \text{and} \quad y \neq 0, \\ 0, & \text{if } x \neq 0 \quad \text{and} \quad y \in \mathbb{Z}. \end{cases}$$

Obviously,

$$\begin{aligned} \text{epi}(f) &= \{(0, 0, z) : z \in [0, \infty)\} \\ &\quad \cup \{(x, y, z) : (x, y) \in (\mathbb{Z} - \{0\}) \times \mathbb{Z} \quad \text{and} \quad z \in [0, \infty)\} \\ &= \overline{\text{epi}(f)}^{\mathbb{Z}^2 \times \mathbb{R}} \end{aligned}$$

but

$$\text{ccl}_{\mathbb{Z}^2 \times \mathbb{R}}(\text{epi}(f)) = \left\{(x, y, z) : (x, y) \in \mathbb{Z}^2 \quad \text{and} \quad z \in [0, \infty)\right\} \neq \overline{\text{epi}(f)}^{\mathbb{Z}^2 \times \mathbb{R}},$$

where $\overline{\text{epi}(f)}^{\mathbb{Z}^2 \times \mathbb{R}}$ indicates the closure of the set epi (f) with respect to subspace topology on $\mathbb{Z}^2 \times \mathbb{R}$. From the previous theorem, we know that f is not convex-l.s.c.

8 Conjugate and Biconjugate of Functions on $\mathbb{Z}^n$

Conjugate and biconjugate (conjugate of conjugate) of functions on $\mathbb{Z}^n$ are keys to duality. In this section, we introduce the corresponding concepts and properties.

Definition 8.1 Let $f : \mathbb{Z}^n \to [-\infty, \infty]$ and $a : \mathbb{R}^n \to \mathbb{R}$. If $a(x) = \langle x, y \rangle + b$, for some $y \in \mathbb{R}^n$ and $b \in \mathbb{R}$, and $f(x) \geqslant a(x)$, $\forall x \in \mathbb{Z}^n$, then the function a is called an affine minorant of function f. The set of all affine minorants of a given function f on $\mathbb{Z}^n$ (possibly being empty) is denoted by Af.

Lemma 8.2 *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$, $Af_{\overline{c}} = Af_c = Af$.

Proof Without loss of generality, we assume that $Af \neq \varnothing$. Since $f_{\overline{c}} \leqslant f_c \leqslant f$, we know $Af_{\overline{c}} \subseteq Af_c \subseteq Af$. For any $a \in Af$, we know that $a \leqslant f$ and a is continuous and convex on $\mathbb{R}^n$. Then, relation (7.10) implies that $a \leqslant f_{\overline{c}}$ and, consequently, $a \in Af_{\overline{c}}$.

Lemma 8.3 *If* $f : \mathbb{Z}^n \to [-\infty, \infty]$ *is convex-l.s.c. on* $\mathbb{Z}^n$, *then*

$$(\mathrm{conv}_{\mathbb{R}^n}(f))_{\overline{c}} = \mathrm{conv}_{\mathbb{R}^n}(f). \tag{8.1}$$

Proof When f is convex-l.s.c., we have $f = f_{\overline{c}}$. Together with (7.6), we see that

$$\mathrm{epi}\,(f) = \mathrm{epi}\,(f_{\overline{c}}) = \mathrm{ccl}_{\mathbb{Z}^n \times \mathbb{R}}\,(\mathrm{epi}\,(f)).$$

Consequently,

$$\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}\,(f)) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{ccl}_{\mathbb{Z}^n \times \mathbb{R}}(\mathrm{epi}\,(f))) = \overline{\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}\,(f))}. \tag{8.2}$$

Combining (7.2), (7.3) and (8.2), we may obtain (8.1).

Recall the next result in conventional convex analysis.

Lemma 8.4 ([15, Lemma 1.40]) *For any* $g : \mathbb{R}^n \to [-\infty, \infty]$, *if* $|g_{\overline{c}}(x_0)| < \infty$ *at some* $x_0 \in \mathbb{R}^n$, *then the set* $Ag \neq \varnothing$.

This lemma leads to the next result.

Corollary 8.5 *For any* $f : \mathbb{Z}^n \to [-\infty, \infty]$, *if* $|f_{\overline{c}}(x_0)| < \infty$ *at some* $x_0 \in \mathbb{Z}^n$, $Af \neq \varnothing$.

Proof Since $f_{\overline{c}}(x_0) < \infty$ at $x_0 \in \mathbb{Z}^n$, by (7.2) and (7.12), we have

$$(\mathrm{conv}_{\mathbb{R}^n}(f))_{\overline{c}}(x_0) = \mathrm{conv}_{\mathbb{R}^n}(f_{\overline{c}})(x_0) = f_{\overline{c}}(x_0) < \infty.$$

The previous lemma implies that $A\mathrm{conv}_{\mathbb{R}}(f) \neq \varnothing$, i.e., there is an affine function a on $\mathbb{R}^n$ such that

$$a(x) \leqslant \mathrm{conv}_{\mathbb{R}}(f)(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Therefore,

$$a(x) \leqslant \mathrm{conv}_{\mathbb{R}}(f)(x) \leqslant f(x) \quad \text{for all} \quad x \in \mathbb{Z}^n,$$

i.e., $a \in Af$.

A direct consequence of (7.11) and [15, Lemma 1.36] gives the next result.

Corollary 8.6 *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$ *with* $\mathrm{dom}\,(f) \neq \varnothing$*,*

$$(\mathrm{conv}_{\mathbb{R}^n}\,(f))_{\overline{c}}\,(x) = \mathrm{conv}_{\mathbb{R}^n}\,(f)\,(x) \quad \text{for all} \quad x \in \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{conv}_{\mathbb{R}^n}\,(f)))\,.$$

Moreover, if $\mathrm{cri}_{\mathbb{Z}^n}\,(\mathrm{dom}\,(f)) \neq \varnothing$*, then*

$$f_{\overline{c}}\,(x) = f_c\,(x) \quad \text{for all} \quad x \in \mathrm{cri}_{\mathbb{Z}^n}\,(\mathrm{dom}\,(f))\,.$$

This leads to the next result.

Lemma 8.7 *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$*, the following conditions are equivalent*:

(*i*) *The set* $Af \neq \varnothing$.
(*ii*) $f_c\,(\cdot) > -\infty$.
(*iii*) $f_{\overline{c}}\,(\cdot) > -\infty$.

Proof If $Af \neq \varnothing$, then, for any $a \in Af$, relation (7.10) implies that $f_{\overline{c}}\,(x) \geqslant a(x)$ for all $x \in \mathbb{Z}^n$. Therefore $(i) \Rightarrow (iii)$. Since $f_{\overline{c}} \leqslant f_c$, $(iii) \Rightarrow (ii)$ is clear. To show $(ii) \Rightarrow (i)$, we assume $f_c(\cdot) > -\infty$. From the first part of the proof of Theorem 6.8, we have

$$-\infty < \mathrm{conv}_{\mathbb{R}^n}\,(f_c) = (\mathrm{conv}_{\mathbb{R}^n}\,(f))_c\,.$$

If $\mathrm{dom}\,(f_c) = \varnothing$, then $f \equiv \infty$ and, clearly, $Af \neq \varnothing$. If $\mathrm{dom}\,(f_c) \neq \varnothing$, then by (6.3)

$$\mathrm{conv}_{\mathbb{R}^n}\,(\mathrm{dom}\,(f_c)) = \mathrm{dom}\,(\mathrm{conv}_{\mathbb{R}^n}\,(f_c)) = \mathrm{dom}\,(\mathrm{conv}_{\mathbb{R}^n}\,(f))\,,$$

which implies that $\mathrm{dom}\,(\mathrm{conv}_{\mathbb{R}^n}\,(f))$ is a nonempty convex set with

$$\mathrm{ri}\,(\mathrm{dom}\,(\mathrm{conv}_{\mathbb{R}^n}\,(f))) \neq \varnothing$$

since every nonempty convex set in $\mathbb{R}^n$ is regular. By Corollary 8.6, for $x_0 \in \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{conv}_{\mathbb{R}^n}\,(f)))$, we have

$$-\infty < (\mathrm{conv}_{\mathbb{R}^n}\,(f))_c\,(x_0) = (\mathrm{conv}_{\mathbb{R}^n}\,(f))_{\overline{c}}\,(x_0) < \infty.$$

Letting $g = \mathrm{conv}_{\mathbb{R}^n}\,(f)$ in Lemma 8.4, we have $A\mathrm{conv}_{\mathbb{R}^n}\,(f) \neq \varnothing$. Thus, $\mathrm{conv}_{\mathbb{R}^n}\,(f) \leqslant f$ on $\mathbb{Z}^n$ and $Af \neq \varnothing$. This completes the proof.

Definition 8.8 $\Gamma\,(\mathbb{Z}^n)$ denotes the set of all proper convex-lower semi-continuous functions $f : \mathbb{Z}^n \to (-\infty, \infty]$, i.e., f is convex-l.s.c. on $\mathbb{Z}^n$ and $f > -\infty$.

Recall the so-called Minkowski's theorem in conventional convex analysis.

Theorem 8.9 *For any function* $g : \mathbb{R}^n \to [-\infty, \infty]$,

$$g \in \Gamma(\mathbb{R}^n) \quad \text{if and only if} \quad Ag \neq \varnothing \quad \text{and} \quad g(x) = \sup\{a(x) : a \in Ag\}.$$

A similar theorem can be deduced for the discrete case.

Theorem 8.10 *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$,

$$f \in \Gamma\left(\mathbb{Z}^n\right) \quad \text{if and only if} \quad Af \neq \varnothing \quad \text{and} \quad f(x) = \sup\{a(x) : a \in Af\}.$$

Proof ($\Rightarrow$) For $f(x) = \sup\{a(x) : a \in Af\}$ with $Af \neq \varnothing$,

$$a(x) \leqslant f_{\bar{c}}(x) \leqslant f(x) \quad \text{for all} \quad a \in Af \quad \text{and} \quad x \in \mathbb{Z}^n.$$

This implies that $f_{\bar{c}}(x) = f(x)$ for all $x \in \mathbb{Z}^m$ and $f > -\infty$, i.e., $f \in \Gamma(\mathbb{Z}^n)$.

($\Leftarrow$) Observe that, for $f \in \Gamma(\mathbb{Z}^n)$, $f_{\bar{c}} = f > -\infty$ and, shown by Lemma 8.7, $Af \neq \varnothing$. Therefore,

$$f(x) \geqslant \sup\{a(x) : a \in Af\} > -\infty.$$

Suppose that $f(x_0) > \sup\{a(x_0) : a \in Af\}$ for some $x_0 \in \mathbb{Z}^n$, then the inclusion relation of $A\mathrm{conv}_{\mathbb{R}^n}(f) \subseteq Af$ implies that

$$\begin{aligned} f(x_0) &> \sup\{a(x_0) : a \in A\mathrm{conv}_{\mathbb{R}^n}(f)\} \\ &= \mathrm{conv}_{\mathbb{R}^n}(f)(x_0) \\ &= f(x_0), \end{aligned}$$

where the first equality is obtained by (8.1) and Theorem 8.9. This leads to a contradiction and completes the proof.

This theorem and Lemma 8.2 lead to the next result.

Corollary 8.11 *If* $f : \mathbb{Z}^n \to [-\infty, \infty]$ *and* $f_{\bar{c}} > -\infty$, *then*

$$Af \neq \varnothing \quad \text{and} \quad f_{\bar{c}}(x) = \sup\{a(x) : a \in Af\}.$$

Proof By the previous theorem, we have $f_{\bar{c}} = \sup\{a(x) : a \in Af_{\bar{c}}\}$ with $Af_{\bar{c}} \neq \varnothing$. The desired result follows from Lemma 8.2.

Definition 8.12 (*Conjugate/biconjugate of functions on* $\mathbb{Z}^n$) For any given function f on $\mathbb{Z}^n$, i.e., $f : \mathbb{Z}^n \to [-\infty, \infty]$, its conjugate is the function $f^* : \mathbb{R}^n \to [-\infty, \infty]$ defined by

$$f^*(y) = \sup\left\{\langle x, y\rangle - f(x) : x \in \mathbb{Z}^n\right\}. \tag{8.3}$$

The function $f^{**}_{\mathbb{Z}^n} : \mathbb{Z}^n \to [-\infty, \infty]$ defined by

$$f^{**}_{\mathbb{Z}^n}(x) = \sup\left\{\langle y, x\rangle - f^*(y) : y \in \mathbb{R}^n\right\} \tag{8.4}$$

is called the biconjugate of function f on $\mathbb{Z}^n$.

By the above definition, it is clear that the conjugate function f^* is convex and lower semi-continuous on $\mathbb{R}^n$. Moreover, if the function $f : \mathbb{Z}^n \to [-\infty, \infty]$ is proper and $Af \neq \varnothing$, then it is easy to prove that f^* is proper.

Remark 8.13 In our definition, for a function on $\mathbb{Z}^n$, its conjugate is a function on $\mathbb{R}^n$ and its biconjugate is a function on $\mathbb{Z}^n$. For example, if $f(x) = \frac{1}{2}x^2$ defined on $\mathbb{Z}$ (the set of integers), then (see [12])

$$f^*(y) = \sup\left\{yx - \frac{1}{2}x^2 : x \in \mathbb{Z}\right\}$$
$$= y\left\lceil y - \frac{1}{2}\right\rceil - \frac{1}{2}\left\lceil y - \frac{1}{2}\right\rceil^2, \quad \forall y \in \mathbb{R}.$$

Moreover,

$$f^{**}_{\mathbb{Z}^n}(x) = \sup\left\{yx - f^*(y) : y \in \mathbb{R}\right\}$$
$$= \frac{1}{2}x^2 = f(x), \quad \forall x \in \mathbb{Z}.$$

The following result is a direct consequence of (8.3).

Lemma 8.14 (Fenchel's inequality) *For any proper function $f : \mathbb{Z}^n \to (-\infty, \infty]$, we have*

$$\langle x, y\rangle \leqslant f(x) + f^*(y) \quad \text{for all} \quad x \in \mathbb{Z}^n \quad \text{and} \quad y \in \mathbb{R}^n.$$

Lemma 8.15 *Let $f : \mathbb{Z}^n \to [-\infty, \infty]$ with $Af \neq \varnothing$, then*

$$(y, r) \in \text{epi}\left(f^*\right) \quad \text{if and only if} \quad a(x) := \langle x, y\rangle - r \in Af.$$

Moreover,

$$f^{**}_{\mathbb{Z}^n}(x) = \sup\{a(x) : a \in Af\} \quad \text{for all} \quad x \in \mathbb{Z}^n. \tag{8.5}$$

Proof Let $a(x) = \langle x, y\rangle - r$ and $a \in Af$, then $a(x) \leqslant f(x)$ for all $x \in \mathbb{Z}^n$, and

$$r \geqslant f^*(y) = \sup\left\{\langle x, y\rangle - f(x) : x \in \mathbb{Z}^n\right\}.$$

Therefore, $(y, r) \in \text{epi}(f^*)$. Conversely, if $(y, r) \in \text{epi}(f^*)$, we know that $r \geqslant f^*(y)$ and, consequently, $a(x) = \langle x, y\rangle - r \leqslant f(x)$ for all $x \in \mathbb{Z}^n$.

To prove (8.5), by the definition of epi (f^*), we know that

$$f^{**}_{\mathbb{Z}^n}(x) = \sup\left\{\langle y, x\rangle - r : (y, r) \in \text{epi}\left(f^*\right)\right\} \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

The first part of the proof leads to

$$f^{**}_{\mathbb{Z}^n}(x) = \sup\{a(x) : a \in Af\} \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

Definition 8.16 (*Convex-closure function of a function on* $\mathbb{Z}^n$) Let $f : \mathbb{Z}^n \to [-\infty, \infty]$ be a function on $\mathbb{Z}^n$, its convex-closure function is defined as $\mathrm{ccl}_{\mathbb{Z}^n}(f) : \mathbb{Z}^n \to [-\infty, \infty]$ with

$$\mathrm{ccl}_{\mathbb{Z}^n}(f) := \begin{cases} f_{\overline{c}}, & \text{if } f_{\overline{c}} > -\infty, \\ -\infty, & \text{otherwise.} \end{cases}$$

Clearly, the function $\mathrm{ccl}_{\mathbb{Z}^n}(f)$ is convex-l.s.c. on $\mathbb{Z}^n$ (see the first part of the proof of Lemma 7.3) and $\mathrm{ccl}_{\mathbb{Z}^n}(f)(\cdot) \leqslant f_{\overline{c}}(\cdot)$.

Theorem 8.17 *For any* $f : \mathbb{Z}^n \to [-\infty, \infty]$*, if* f *is convex on* $\mathbb{Z}^n$*, then*

$$(\mathrm{ccl}_{\mathbb{Z}^n}(f))^*(x) = f^*(x) \quad \text{for all} \quad x \in \mathbb{Z}^n. \tag{8.6}$$

Proof If $f_{\overline{c}} > -\infty$, then we have $(\mathrm{ccl}_{\mathbb{Z}^n}(f))^* = (f_{\overline{c}})^* \geqslant f^*$, and $Af \neq \varnothing$ by Lemma 8.7. If $(f_{\overline{c}})^*(y_0) > f^*(y_0)$ for some $y_0 \in \mathbb{R}^n$, then we would have $(y_0, f^*(y_0)) \notin \mathrm{epi}(f_{\overline{c}})$, and hence,

$$\langle x, y_0 \rangle - f^*(y_0) \notin Af_{\overline{c}} = Af$$

by Lemmas 8.2 and 8.15. This means, $\langle x_0, y_0 \rangle - f^*(y_0) > f(x_0)$ for some $x_0 \in \mathbb{Z}^n$, which is impossible since $f > -\infty$ (see Lemma 8.14). Hence, we have

$$(\mathrm{ccl}_{\mathbb{Z}^n}(f))^* = (f_{\overline{c}})^* = f^*.$$

If $f_{\overline{c}} > -\infty$ is not true, then by Lemma 8.7 we know that $f_c(\hat{x}_0) = -\infty$ for some $\hat{x}_0 \in \mathbb{Z}^n$. Since f is convex, we have $f_c(\hat{x}_0) = f(\hat{x}_0) = -\infty$. This along with (8.3) yields $f^* = +\infty = (\mathrm{ccl}_{\mathbb{Z}^n}(f))^*$. This completes the proof.

Definition 8.18 Let $f : \mathbb{Z}^n \to [-\infty, \infty]$ be a function. f is integrally closed on $\mathbb{Z}^n$ if

$$f(x) = \mathrm{ccl}_{\mathbb{Z}^n}(f)(x) \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

Remark 8.19 If $f : \mathbb{Z}^n \to (-\infty, \infty]$ is a proper convex function, then, by (7.9) and Lemma 8.7, we have

$$\mathrm{ccl}_{\mathbb{Z}^n \times \mathbb{R}}(\mathrm{epi}(f)) = \mathrm{epi}(f_{\overline{c}}) = \mathrm{epi}(\mathrm{ccl}_{\mathbb{Z}^n}(f)).$$

Theorem 7.6 implies that, for a proper convex function, the integrally closeness is equivalent to convex-lower semi continuity. However, the integrally closed improper functions are the constant functions of $-\infty$ and ∞ only.

Theorem 8.20 (Fenchel–Moreau theorem on $\mathbb{Z}^n$) *For any function* $f : \mathbb{Z}^n \to [-\infty, \infty]$,

$$f^{**}_{\mathbb{Z}^n}(x) = \text{ccl}_{\mathbb{Z}^n}(f)(x) \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

Proof (i) If $f_{\bar{c}}(x_0) = -\infty$ for some $x_0 \in \mathbb{Z}^n$, then we claim that $f^* \equiv \infty$. If this is not true, i.e., $f^*(y_0) < \infty$ at some $y_0 \in \mathbb{R}^n$, then there exists some $r \in \mathbb{R}$ such that

$$r \geqslant \langle x, y_0 \rangle - f(x) \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

Consequently, the function $a(x) = \langle x, y_0 \rangle - r \in Af$. By Lemma 8.7, we know that $f_{\bar{c}} > -\infty$ which causes a contradiction. Now, since $f^* \equiv \infty$, we know $f^{**}_{\mathbb{Z}^n} \equiv -\infty$. By Definition 8.16, we have $f^{**}_{\mathbb{Z}^n} = \text{ccl}_{\mathbb{Z}^n}(f)$.

(ii) If $f_{\bar{c}}(x) > -\infty$ for all $x \in \mathbb{Z}^n$, then by Theorem 8.10 we have $Af \neq \varnothing$ and

$$f_{\bar{c}}(x) = \sup\{a(x) : a \in Af\}.$$

From Lemmas 8.15, 8.7 and (8.5), we have

$$\begin{aligned} f^{**}_{\mathbb{Z}^n}(x) &= \sup\{a(x) : a \in Af\} \\ &= f_{\bar{c}}(x) \\ &= \text{ccl}_{\mathbb{Z}^n}(f)(x) \quad \text{for all} \quad x \in \mathbb{Z}^n. \end{aligned}$$

Lemma 8.21 *Let* $f : \mathbb{Z}^n \to (-\infty, \infty]$ *be a proper convex-l.s.c. and positively homogeneous function on* $\mathbb{Z}^n$ *and* $C = \{y \in \mathbb{R}^n : f^*(y) \leqslant 0\}$. *Then, C is a nonempty closed convex set in* $\mathbb{R}^n$ *and*

$$f(x) = \sup\{\langle x, y \rangle : y \in C\} \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

Proof By Theorem 8.20, we have

$$f(x) = f^{**}_{\mathbb{Z}^n}(x) = \sup_{y \in \mathbb{R}^n} \left\{\langle x, y \rangle - f^*(y)\right\} \quad \text{for all} \quad x \in \mathbb{Z}^n. \tag{8.7}$$

Since f is positively homogeneous,

$$\alpha f^*(y) = \sup_{x \in \mathbb{Z}^n} \{\langle \alpha x, y \rangle - f(\alpha x)\} = f^*(y)$$

for all $\alpha \in \mathbb{N}$ and $y \in \mathbb{R}^n$. This implies that

$$f^*(y) \in \{-\infty, 0, \infty\}. \tag{8.8}$$

If $f^*(y) = \infty$, then $f^{**}_{\mathbb{Z}^n}(x) = -\infty$ for all $x \in \mathbb{Z}^n$. By (8.7), we have $f = -\infty$, which contradicts the assumption of $f > -\infty$. Therefore, f^* is not identical to ∞

and (8.8) implies that $C \neq \varnothing$. Again, by (8.7), we have

$$f(x) = f^{**}_{\mathbb{Z}^n}(x) = \sup_{y \in \mathbb{R}^n} \left\{ \langle x, y \rangle - f^*(y) \right\} = \sup_{y \in C} \langle x, y \rangle \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

Since the function f^* is convex and l.s.c. on $\mathbb{R}^n$, C is a closed and convex subset of $\mathbb{R}^n$.

Definition 8.22 (*Subgradient of a function on* $\mathbb{Z}^n$) Let $f : \mathbb{Z}^n \to [-\infty, \infty]$ and $x_0 \in \mathbb{Z}^n$, the set

$$\partial f_{\mathbb{Z}^n}(x_0) = \left\{ y_0 \in \mathbb{R}^n : f(x) \geqslant f(x_0) + \langle x - x_0, y_0 \rangle \quad \text{for all} \quad x \in \mathbb{Z}^n \right\}$$

is called the subgradient set of f at the point $x_0 \in \mathbb{Z}^n$.

Lemma 8.23 *Let* $f : \mathbb{Z}^n \to [-\infty, \infty]$ *with* $|f(x_0)| < \infty$ *at some* $x_0 \in \mathbb{Z}^n$*, then*

$$y_0 \in \partial f_{\mathbb{Z}^n}(x_0) \quad \text{if and only if} \quad f(x_0) + f^*(y_0) = \langle x_0, y_0 \rangle .$$

Proof If $y_0 \in \partial f_{\mathbb{Z}^n}(x_0)$, by definition, $f(x) \geqslant f(x_0) + \langle x - x_0, y_0 \rangle$ for all $x \in \mathbb{Z}^n$. Since $-\infty < f(x_0) < \infty$, we have $\langle x_0, y_0 \rangle - f(x_0) \geqslant \langle x, y_0 \rangle - f(x)$ for all $x \in \mathbb{Z}^n$. By the definition of $f^*(y_0)$, we have $\langle x_0, y_0 \rangle - f(x_0) = f^*(y_0)$.

If $f(x_0) + f^*(y_0) = \langle x_0, y_0 \rangle$ holds for some $y_0 \in \mathbb{R}^n$, then (8.3) implies that

$$\begin{aligned} f(x_0) - \langle x_0, y_0 \rangle &= -f^*(y_0) \\ &\leqslant f(x) - \langle x, y_0 \rangle \quad \text{for all} \quad x \in \mathbb{Z}^n. \end{aligned}$$

Thus we have $y_0 \in \partial f_{\mathbb{Z}^n}(x_0)$.

Recall the next result in conventional convex analysis.

Theorem 8.24 ([15, Theorem 1.13]) *If the function* $g : \mathbb{R}^n \to [-\infty, \infty]$ *is convex and* $-\infty < g(x_0) < \infty$ *at some* $x_0 \in \mathrm{ri}(\mathrm{dom}\, f)$*, then* $\partial g(x_0) \neq \varnothing$.

A similar result holds for the functions on $\mathbb{Z}^n$.

Theorem 8.25 *If function* $f : \mathbb{Z}^n \to [-\infty, \infty]$ *is convex on* $\mathbb{Z}^n$ *and* $-\infty < f(x_0) < \infty$ *at some* $x_0 \in \mathrm{cri}_{\mathbb{Z}^n}(\mathrm{dom}\, f)$*, then* $\partial f_{\mathbb{Z}^n}(x_0) \neq \varnothing$.

Proof Since $\mathrm{cri}_{\mathbb{Z}^n}(\mathrm{dom}\, f) = \mathrm{ri}(\mathrm{dom}(\mathrm{conv}_{\mathbb{R}^n}(f))) \cap \mathbb{Z}^n$, letting $g = \mathrm{conv}_{\mathbb{R}^n}(f)$ in the previous theorem, we know the existence of $y_0 \in \mathbb{R}^n$ such that

$$\mathrm{conv}_{\mathbb{R}^n}(f)(x) \geqslant f(x_0) + \langle x - x_0, y_0 \rangle \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Noticing that $\mathrm{conv}_{\mathbb{R}^n}(f)|_{\mathbb{Z}^n} = f$, we have

$$f(x) \geqslant f(x_0) + \langle x - x_0, y_0 \rangle \quad \text{for all} \quad x \in \mathbb{Z}^n.$$

Consequently, $\partial f_{\mathbb{Z}^n}(x_0) \neq \varnothing$.

9 Optimization and Duality

In this section, we show how the results of convex analysis on discrete domains can be used for solving optimization problems. In particular, we introduce a dual problem of the optimization problem on a discrete domain.

9.1 Duality Theory

Let $f : \mathbb{Z}^n \to [-\infty, \infty]$ be an arbitrary function and consider the following primal optimization problem:

$$v(P) := \inf \left\{ f(x) : x \in \mathbb{Z}^n \right\}. \tag{P}$$

We let m be a positive integer, and we associate the function f with a function $F : \mathbb{Z}^n \times \mathbb{Z}^m \to [-\infty, \infty]$ satisfying $F(x, \mathbf{0}) = f(x)$ for all $x \in \mathbb{Z}^n$. Consider a perturbation function $p : \mathbb{Z}^m \to [-\infty, \infty]$ defined by

$$p(y) := \inf \left\{ F(x, y) : x \in \mathbb{Z}^n \right\}. \tag{9.1}$$

Obviously, we see that

$$p(\mathbf{0}) = v(P).$$

Using the conjugate function of $p(\cdot)$, we define a dual problem of (P) as below.

Definition 9.1 [*Dual problem of* (P)] The dual problem of problem (P) is

$$v(D) := \sup \left\{ -p^*(\omega) : \omega \in \mathbb{R}^m \right\}, \tag{D}$$

where p^* is the conjugate function of p defined in Definition 8.12.

Observe that Definition 8.12 along with (D) yields

$$p^{**}_{\mathbb{Z}^m}(\mathbf{0}) = \sup_{\omega \in \mathbb{R}^m} \left(-p^*(\omega) \right) = v(D). \tag{9.2}$$

Since

$$\begin{aligned} p(\mathbf{0}) &= -\{\langle \omega, 0 \rangle - p(0)\} \\ &\geqslant -\sup_{y \in \mathbb{Z}^m} \{\langle \omega, y \rangle - p(y)\} \\ &= -p^*(\omega) \end{aligned}$$

for all $\omega \in \mathbb{R}^m$, we have

$$p(\mathbf{0}) \geqslant \sup_{\omega \in \mathbb{R}^m} \left\{ -p^*(\omega) \right\} = p^{**}_{\mathbb{Z}^m}(\mathbf{0}).$$

Putting things together with (9.2), we have the following weak duality property:

$$v\,(D) = p^{**}_{\mathbb{Z}^m}\,(\mathbf{0}) \leqslant p\,(\mathbf{0}) = v\,(P)\,. \tag{9.3}$$

We would like to figure out some conditions for the equality $v\,(P) = v\,(D)$ to hold. Notice that if $v\,(P) = -\infty$, then (9.3) implies that $v\,(D) = -\infty$ and every $\omega \in \mathbb{R}^m$ is an optimal solution to the dual problem (D). Therefore, we only need to investigate the case of $v\,(P) > -\infty$. The case $v(P) = \infty$ can be illustrated similar to [15, Example 1.16]. Hereafter, we study the case of $v\,(P) < \infty$.

Theorem 9.2 *Assume that function $p : \mathbb{Z}^m \to [-\infty, \infty]$ is convex and $p\,(\mathbf{0}) = v\,(P) < \infty$. Then, we have*

$$v\,(P) = v\,(D) \quad \text{if and only if} \quad p \text{ is convex-l.s.c. at } \mathbf{0}.$$

Proof If p is convex-l.s.c. at $\mathbf{0}$, then

$$p_{\overline{c}}\,(\mathbf{0}) = p\,(\mathbf{0}) < \infty,$$

and $Ap \neq \varnothing$ by Corollary 8.5. Using Lemma 8.7, we have $p_{\overline{c}} > -\infty$. Along with Definition 8.16, we see

$$\operatorname{ccl}_{\mathbb{Z}^m}\,(p)\,(\mathbf{0}) = p_{\overline{c}}\,(\mathbf{0})\,.$$

Fenchel–Moreau theorem (Theorem 8.20) implies that

$$v\,(P) = p\,(\mathbf{0}) = p_{\overline{c}}\,(\mathbf{0}) = \operatorname{ccl}_{\mathbb{Z}^m}\,(p)\,(\mathbf{0}) = p^{**}_{\mathbb{Z}^m}\,(\mathbf{0}) = v\,(D)\,.$$

Conversely, if $v\,(P) = v\,(D)$, then

$$p\,(\mathbf{0}) = v\,(P) = v\,(D) = p^{**}_{\mathbb{Z}^m}\,(\mathbf{0}) = \operatorname{ccl}_{\mathbb{Z}^m}\,(p)\,(\mathbf{0})\,,$$

which shows that $\operatorname{ccl}_{\mathbb{Z}^m}\,(p)\,(\mathbf{0})$ is finite. It follows from Definition 8.16 that $\operatorname{ccl}_{\mathbb{Z}^m}\,(p)\,(\mathbf{0}) = p_{\overline{c}}\,(\mathbf{0}) = p\,(\mathbf{0})$ and $p(\cdot)$ is convex-l.s.c. at $\mathbf{0}$. This completes the proof.

The above result immediately leads to a sufficient condition for zero duality gap in the next.

Corollary 9.3 *Assume that function $p : \mathbb{Z}^m \to [-\infty, \infty]$ is convex and $p\,(\mathbf{0}) = v\,(P) < \infty$. If*

$$\mathbf{0} \in \operatorname{cri}_{\mathbb{Z}^m}\,(\operatorname{dom}\,(p))\,,$$

then the dual problem (D) has an optimal solution and $v\,(P) = v\,(D)$.

Proof If $\mathbf{0} \in \mathrm{cri}_{\mathbb{Z}^m}(\mathrm{dom}\,(p))$ and $p\,(\mathbf{0})$ is finite, then Theorem 8.25 says that $\partial p_{\mathbb{Z}^n}(\mathbf{0}) \neq \varnothing$. By Lemma 8.23, it is easy to verify that any $\omega_0 \in \partial p_{\mathbb{Z}^n}(\mathbf{0})$ is an optimal solution of the dual problem. Furthermore, Corollary 8.6 leads to $p\,(\mathbf{0}) = p_{\overline{c}}\,(\mathbf{0})$. The preceding theorem implies that $v\,(P) = v\,(D)$, which completes the proof.

Let $f : \mathbb{Z}^n \to \mathbb{R}$ be a real-valued function and $g : \mathbb{Z}^n \to \mathbb{Z}^m$ a vector valued function represented by $g\,(x) := (g_1\,(x)\,, g_2\,(x)\,, \cdots, g_m(x))$, where $g_i : \mathbb{Z}^n \to \mathbb{Z}$ for $i = 1, 2, \cdots, m$. Consider the special case of problem (P) given by

$$\inf_{x\in D} \{f\,(x) : g\,(x) \in -K\}\,, \tag{P1}$$

where $D \subseteq \mathbb{Z}^n$ is a nonempty set, and $K \subseteq \mathbb{Z}^m$ is a convex cone in $\mathbb{Z}^m$ such that

$$\mathbf{0} \in K \cap g\,(D)\,. \tag{9.4}$$

We now derive an analogue of the Lagrangian perturbation scheme for the optimization problem (P1) over discrete domains. First, define a function $F : \mathbb{Z}^n \times \mathbb{Z}^m \to [-\infty, \infty]$ by

$$F\,(x, y) = \begin{cases} f\,(x)\,, & \text{if } x \in D \quad \text{and} \quad g\,(x) \in -K + y, \\ \infty, & \text{otherwise.} \end{cases}$$

For this specific choice of F, by (9.1), we see that

$$p\,(y) = \inf_{x\in D} \{f\,(x) : g\,(x) \in -K + y\}\,. \tag{9.5}$$

Note that the problem (P1) actually includes some important classes of optimization problems shown in the following example:

Example 9.4
1. For $f(x) = c^{\mathrm{T}}x$ and $g\,(x) = Ax - b$ with A being an $m \times n$ matrix, $K = \{\mathbf{0}\}$ and $D = \mathbb{Z}^n$, the optimization problem (P1) reduces to the following integer linear programming problem on a discrete domain:

$$\inf \left\{c^{\mathrm{T}}x : Ax = b, x \in \mathbb{Z}^n\right\}.$$

2. If $m = n$ and $g\,(x) = -x$, then the problem (P1) reduces to the following generalized geometric programming problem on a discrete domain:

$$\inf \{f\,(x) : x \in K \cap D\}\,.$$

3. If the nonempty convex cone $K \subseteq \mathbb{Z}^m$ is chosen as $K = \mathbb{Z}_+^p \times \{\mathbf{0}\}$ with $\mathbf{0} \in \mathbb{Z}^{m-p}$ for some $p \leqslant m$, and $D = \mathbb{Z}^n$, then the problem (P1) reduces to the following nonlinear programming problems on a discrete domain:

$$\inf \{f\,(x) : g_i\,(x) \leqslant 0, \quad i = 1, 2, \cdots, p, \quad g_i\,(x) = 0, \\ p + 1 \leqslant i \leqslant m, \quad x \in D\}\,.$$

Using the representation of $p(\cdot)$ in (9.5), we can provide more detailed expressions of the dual problem commonly called Lagrangian dual.

Lemma 9.5 *Let K be a convex cone satisfying (9.4), D a nonempty set,*

$$\mathcal{H}\left(\mathbb{Z}^m\right) := \left\{y \in \mathbb{Z}^m : D \cap g^{-1}\{-K + y\} \neq \varnothing\right\}$$

and $\gamma : \mathbb{R}^m \to \mathbb{R}$ a function defined by

$$\gamma(\omega) := \inf_{y \in \mathcal{H}(\mathbb{Z}^m)} \left\{\inf\left\{f(x) - \langle \omega, y\rangle : x \in D \cap g^{-1}\{-K + y\}\right\}\right\}.$$

Then

$$\gamma(w) = \begin{cases} \inf_{x \in D}\{f(x) - \langle \omega, g(x)\rangle\}, & \text{if } \omega \in -K^*, \\ -\infty, & \text{if } \omega \notin -K^*. \end{cases}$$

Proof Suppose that (9.4) holds. One may easily verify that

$$g(D) \subset \mathcal{H}\left(\mathbb{Z}^m\right) \quad \text{and} \quad K \subset \mathcal{H}\left(\mathbb{Z}^m\right).$$

We then have

$$\begin{aligned} \gamma(w) &= \inf_{y \in \mathcal{H}(\mathbb{Z}^m)} \{\inf\{f(x) - \langle \omega, y\rangle : x \in D \quad \text{and} \quad y = g(x) + k \quad \text{for } k \in K\}\} \\ &= \inf_{k \in K} \{\inf\{f(x) - \langle \omega, g(x) + k\rangle : x \in D \quad \text{and} \\ &\qquad k = y - g(x) \quad \text{for} \quad y \in \mathcal{H}\left(\mathbb{Z}^m\right)\}\}. \end{aligned} \tag{9.6}$$

If $\omega \in -K^*$, then $\langle \omega, k\rangle \leqslant 0$ for all $k \in K$. Since $g(D) \subset \mathcal{H}(\mathbb{Z}^m)$, there exists $x_0 \in D$ such that $y_0 = g(x_0) \in \mathcal{H}(\mathbb{Z}^m)$. This means that we can choose $k = 0$ in the second infimum in (9.6) to get $\gamma(\omega) = \inf_{x \in D}\{f(x) - \langle \omega, g(x)\rangle\}$.

If $\omega \notin -K^*$, then one can find a $k_0 \in K$ such that $\langle \omega, k_0\rangle > 0$ holds. Since K is a cone in $\mathbb{Z}^m$, we have $\alpha k_0 \in K \subset \mathcal{H}(\mathbb{Z}^m)$ for all $\alpha \in \mathbb{N}$. Since $0 \in g(D)$, the first infimum in (9.6) becomes $-\infty$.

Theorem 9.6 [Lagrangian dual of problem (P1)] *Consider problem (P1) with the cone K satisfying (9.4). When the function $\Theta : -K^* \to [-\infty, \infty]$ is defined by*

$$\Theta(\omega) := \inf_{x \in D}\{f(x) - \langle \omega, g(x)\rangle\},$$

the Lagrangian dual of optimization problem (P1) becomes

$$v(\text{LD}) := \sup\left\{\Theta(\omega) : \omega \in -K^*\right\}. \tag{LD1}$$

Proof By the definition of the function p, for every $\omega \in \mathbb{R}^m$, we have

$$\begin{aligned}
-p^*(\omega) &= -\sup_{y\in\mathbb{Z}^m}\{\langle\omega, y\rangle - p(y)\} \\
&= -\sup_{y\in\mathbb{Z}^m}\{\langle\omega, y\rangle - \inf\{f(x) : x \in D \quad \text{and} \quad g(x) \in -K + y\}\} \\
&= -\sup_{y\in\mathbb{Z}^m}\{\sup\{\langle\omega, y\rangle - f(x) : x \in D \quad \text{and} \quad y \in g(x) + K\}\} \\
&= \inf_{y\in\mathbb{Z}^m}\left\{\inf_{x\in D}\{f(x) - \langle\omega, y\rangle : x \in D \quad \text{and} \quad y \in g(x) + K\}\right\} \\
&= \gamma(\omega). \qquad (9.7)
\end{aligned}$$

The rest of the proof follows from the preceding lemma.

The following is an example for illustration.

Example 9.7 Let $f(x) = \left(x - \frac{1}{2}\right)^2$, $K = \{0\}$, $g(x) = 3x - 6$ and $D = \mathbb{Z}$. Clearly (9.4) holds and (P1) becomes

$$v(P) = \inf\left\{\left(x - \frac{1}{2}\right)^2 : 3x - 6 = 0 \quad \text{and} \quad x \in \mathbb{Z}\right\},$$

which has an optimal solution $x = 2$ with the value $\frac{9}{4}$. For this choice of (P1), we have

$$\begin{aligned}
-p^*(\omega) &= \inf\{f(x) - \langle\omega, g(x)\rangle : x \in D\} \\
&= \inf_{x\in\mathbb{Z}}\left\{\left(x - \frac{1}{2}\right)^2 - \omega(3x - 6)\right\} \\
&= \inf_{x\in\mathbb{Z}} \chi(x, \omega),
\end{aligned}$$

where $\chi(x, \omega) := \left(x - \frac{1}{2}\right)^2 - \omega(3x - 6)$. To find the points $x \in \mathbb{Z}$ that minimize the function χ, we need to solve

$$\nabla\chi(x, \omega) \leqslant 0 \leqslant \Delta\chi(x, \omega),$$

(see [12]) or, equivalently,

$$2x - 3\omega - \frac{1}{2} \leqslant 0 \leqslant 2x - 3\omega,$$

where ∇ and Δ are the backward and the forward difference operators, respectively. Then, we have

$$-p^*(\omega) = \chi\left(\frac{1}{2}\lceil 3\omega\rceil, \omega\right) = \left(\left(\frac{1}{2}\lceil 3\omega\rceil - \frac{1}{2}\right)^2 - \omega\left(\frac{3}{2}\lceil 3\omega\rceil - 6\right)\right),$$

where $\lceil x \rceil$ stands for the smallest integer greater than x. The preceding theorem leads to the following dual Lagrangian problem:

$$v\left(LD\right) := \sup_{\omega \in \mathbb{R}} \left\{ \left(\left(\frac{1}{2} \lceil 3\omega \rceil - \frac{1}{2} \right)^2 - \omega \left(\frac{3}{2} \lceil 3\omega \rceil - 6 \right) \right) \right\}.$$

Obviously, this problem has an optimal solution $\omega = \frac{1}{2}$ with the optimal value

$$v\left(P\right) = v\left(\mathrm{LD}\right) = \frac{9}{4}.$$

Theorem 9.8 *For the primal problem* (P1)*, let the vector valued function* $h : \mathbb{Z}^n \to \mathbb{Z}^m \times \mathbb{R}$ *be*

$$h\left(x\right) := \left(g\left(x\right), f\left(x\right)\right). \tag{9.8}$$

If $h\left(D\right) + \left(K \times (0, \infty)\right)$ *is convex in* $\mathbb{Z}^m \times \mathbb{R}$ *and*

$$\mathbf{0} \in \mathrm{cri}_{\mathbb{Z}^m}\left(g\left(D\right) + K\right), \tag{9.9}$$

then $\infty > v\left(P1\right) = v\left(LD1\right)$ *and the Lagrangian dual problem* ($LD1$) *has an optimal solution.*

Proof It is straightforward to show that

$$\widetilde{\mathrm{epi}\left(p\right)} = h\left(D\right) + \left(K \times (0, \infty)\right),$$

where $\widetilde{\mathrm{epi}\left(p\right)}$ is given by (4.5). By Corollary 4.19, we know that the function p is convex. Since $\mathrm{dom}\left(p\right) = g\left(D\right) + K$, condition (9.9) along with Corollary 9.3 and the preceding theorem yields the desired result.

9.2 Dual of Integer Linear Programming (ILP) Problem

In this subsection, we explicitly address a dual approach for solving integer linear programming problems.

Consider the following integer linear programming (ILP) problem:

$$v(\mathrm{ILP}) := \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{Z}_+^n \quad \text{and} \quad Ax - b \leqslant \mathbf{0} \right\} \tag{ILP}$$

where $c \in \mathbb{R}^n$, A is an $m \times n$ matrix and b an m-vector. This is a special case of the problem (P1) with $D = \mathbb{Z}_+^n$, $K = \mathbb{Z}_+^m$, and $g\left(x\right) := Ax - b$. Here $\mathbb{Z}_+$ means the set of nonnegative integers. To establish the dual problem using the convex analysis arguments developed in the previous sections, we first need to make sure that

$$g\left(\mathbb{Z}_+^n\right) = \left(A\left(\mathbb{Z}_+^n\right) - b\right) \subset \mathbb{Z}^m,$$

which requires the constraint function $g(x) = Ax - b$ to be integer valued. If the constraint function g is not integer valued, then we may restate the problem (ILP) in terms of a new constraint function $g^+(\cdot) : \mathbb{Z}^n \to \mathbb{Z}^m$ with $g^+(x) := \lceil Ax - b \rceil$ for all $x \in \mathbb{Z}^n_+$, where $\lceil Ax - b \rceil \in \mathbb{Z}^m$ is the vector with components that are least integers greater than those of $Ax - b$. Also denote that $g^+\left(\mathbb{Z}^n_+\right) := \left\lceil A\left(\mathbb{Z}^n_+\right) - b \right\rceil$.

Notice that, for all $x \in \mathbb{Z}^n_+$,

$$Ax - b \leqslant \mathbf{0} \quad \text{if and only if} \quad g^+(x) := \lceil Ax - b \rceil \leqslant \mathbf{0}. \tag{9.10}$$

Thus, problem (ILP) is equivalent to the following:

$$v^+(\text{ILP}) := \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{Z}^n_+ \quad \text{and} \quad \lceil Ax - b \rceil \leqslant \mathbf{0} \right\}. \tag{9.11}$$

Let us assume that

$$\mathbf{0} \in g^+\left(\mathbb{Z}^n_+\right) \cap \mathbb{Z}^m_+. \tag{9.12}$$

Then, we can construct a dual of (9.11) by using the arguments in the preceding sections. In the rest of this subsection, the dual of (ILP) actually means the dual of (9.11).

By substituting

$$\begin{aligned} \Theta(\omega) &= \inf_{x \in \mathbb{Z}^n_+} \left\{ c^{\mathrm{T}} x - \left\langle \omega, g^+(x) \right\rangle \right\} \\ &= \inf_{x \in \mathbb{Z}^n_+} \left\{ c^{\mathrm{T}} x - \left\langle \omega, \lceil Ax - b \rceil \right\rangle \right\} \end{aligned}$$

into (LD1), we have the following dual problem of (9.11):

$$v^+(\text{DILP}) := \sup_{\omega \in \mathbb{R}^m_-} \left\{ \omega^{\mathrm{T}} b + R(\omega) \right\}, \tag{9.13}$$

where

$$R(\omega) := \inf_{x \in \mathbb{Z}^n_+} \left\{ \left(c - A^{\mathrm{T}} \omega \right)^{\mathrm{T}} x - \omega^{\mathrm{T}} \left(\lceil Ax - b \rceil - (Ax - b) \right) \right\} \quad \text{for} \quad \omega \in \mathbb{R}^m_- \tag{9.14}$$

and $\mathbb{R}^m_-$ is the set of vectors with nonpositive real components.

Lemma 9.9 *Let $R(\omega)$ be defined by (9.14), then for $b \in \mathbb{Z}^m$ and $\omega \in \mathbb{R}^m_-$, we have*

$$R(w) = \begin{cases} 0, & \text{if } c - A^{\mathrm{T}} \omega \in \mathbb{R}^m_+, \\ -\infty, & \text{otherwise.} \end{cases} \tag{9.15}$$

Proof Let $b \in \mathbb{Z}^m$ and $\omega \in \mathbb{R}^m_-$. If $c - A^{\mathrm{T}}\omega \in \mathbb{R}^m_+$, then

$$\left(c - A^{\mathrm{T}}\omega\right)^{\mathrm{T}} x - \omega^{\mathrm{T}} \left(\lceil Ax - b \rceil - (Ax - b)\right) \geqslant 0$$

for all $x \in \mathbb{Z}^n_+$. Since

$$\omega^{\mathrm{T}} \left(\lceil Ax - b \rceil - (Ax - b)\right) = 0 \quad \text{for } x = 0 \quad \text{and} \quad b \in \mathbb{Z}^m,$$

we obtain $R(\omega) = 0$.

If $c - A^{\mathrm{T}}\omega < \mathbf{0}$ (i.e., if all entries of $c - A^{\mathrm{T}}\omega$ are negative), then

$$\inf \left\{ \left(c - A^{\mathrm{T}}\omega\right)^{\mathrm{T}} x : x \in \mathbb{Z}^n_+ \right\} = -\infty. \tag{9.16}$$

If $R(\omega) > -\infty$, then

$$\begin{aligned} \left(c - A^{\mathrm{T}}\omega\right)^{\mathrm{T}} x &\geqslant R(\omega) + \omega^{\mathrm{T}} \left(\lceil Ax - b \rceil - (Ax - b)\right) \\ &\geqslant R(\omega) + \omega^{\mathrm{T}}\mathbf{1} \end{aligned}$$

for all $x \in \mathbb{Z}^n_+$, where $\mathbf{1}$ is the vector whose components are all 1. Together with (9.16), we have the desired result.

Using (9.13), we immediately have the next result.

Theorem 9.10 (Lagrangian dual of (ILP) with integral b) *Let $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{Z}^m$ such that $\mathbf{0} \in g^+\left(\mathbb{Z}^n_+\right) \cap \mathbb{Z}^m_+$. Then, the Lagrangian dual of* (ILP) *becomes*

$$v\,(\text{DILP}) = \sup_{\omega \in \mathbb{R}^m_-} \left\{ \omega^{\mathrm{T}} b : c - A^{\mathrm{T}}\omega \geqslant \mathbf{0} \right\}. \tag{DILP}$$

For an integral matrix $A \in \mathbb{Z}^{m \times n}$, the function $g^+(x) = \lceil Ax - b \rceil$ turns into $g^+(x) = Ax - \lfloor b \rfloor$. With a similar procedure to that of [15, Lemma 1.57] and combining (9.13) and (9.14), it is straightforward to prove the next result.

Theorem 9.11 *(*Lagrangian dual of (ILP) with integral matrix A*) Assume that $A \in \mathbb{Z}^{m \times n}$ is an integral matrix and $b \in \mathbb{R}^m$ such that $\mathbf{0} \in g^+\left(\mathbb{Z}^n_+\right) \cap \mathbb{Z}^m_+$. If there exists $x \in \mathbb{Z}^n_+$ such that $Ax = \lfloor b \rfloor$, then the primal problem* (*ILP*) *is equivalent to*

$$v(\text{ILP}) = \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{Z}^n_+ \quad \text{and} \quad Ax \leqslant \lfloor b \rfloor \right\}$$

and the Lagrangian dual of (ILP) *becomes*

$$v\,(\text{DILP}) = \sup_{\omega \in \mathbb{R}^m_-} \left\{ \omega^{\mathrm{T}} \lfloor b \rfloor : c - A^{\mathrm{T}}\omega \geqslant \mathbf{0} \right\},$$

where $\lfloor b \rfloor$ is component wise greatest integer of b.

As a direct consequence of Theorem 9.8, we may derive a discrete version of Slater's condition to ensure the strong duality theorem for ILP problems in the next result.

Theorem 9.12 (Strong duality for ILP) *Given* $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^m$ *and* $c \in \mathbb{R}^n$, *consider the following primal problem*:

$$v(\text{ILP}) := \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{Z}_+^n \quad \text{and} \quad Ax - b \leqslant \mathbf{0} \right\}.$$

Assume that $A = \left[a_{ij}\right]_{m\times n}$ *and* $b = [b_i]_{m\times 1}$ *such that the function* $g^+(x) = \lceil Ax - b \rceil$ *satisfies* $\mathbf{0} \in g^+\left(\mathbb{Z}_+^n\right) \cap \mathbb{Z}_+^m$ *and the function*

$$p_{\text{ILP}}(y) := \inf_{x\in\mathbb{Z}_+^n} \left\{ c^{\mathrm{T}} x : g^+(x) \leqslant y \right\} \tag{9.17}$$

is convex on $\mathbb{Z}^m$ *or, equivalently, the region* $h_{\text{ILP}}\left(\mathbb{Z}_+^n\right) + \left(\mathbb{Z}_+^m \times (0,\infty)\right)$ *is convex in* $\mathbb{Z}^m \times \mathbb{R}$, *where*

$$h_{\text{ILP}}(x) := \left(c^{\mathrm{T}} x, g^+(x) \right).$$

If

$$\mathbf{0} \in \text{ri}\left(\text{conv}_{\mathbb{R}^n}\left(g^+\left(\mathbb{Z}_+^n\right) + \mathbb{Z}_+^m\right)\right), \tag{9.18}$$

then

$$\infty > v(\text{ILP}) = v(\text{DILP}).$$

Theorem 9.13 (Convexity and strong duality) *Let* $A \in \mathbb{Z}^{m\times n}$ *be an integral matrix and* $b \in \mathbb{R}^m$ *such that the function* p_{ILP} *defined by* (9.17) *is convex on* $\mathbb{Z}^m$. *Consider the following primal problem*:

$$v(\text{ILP}) := \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{Z}_+^n \quad \text{and} \quad Ax - b \leqslant \mathbf{0} \right\}.$$

If there exist vectors $x^0 \in \mathbb{Z}_+^n$ *and* $x^1 \in (0,\infty)^n$ *such that* $Ax^0 = \lfloor b \rfloor$ *and* $Ax^1 < \lfloor b \rfloor$, *then we have*

$$\infty > v(\text{ILP}) = v(\text{DILP}) = \sup_{\omega\in\mathbb{R}_-^m} \left\{ \omega^{\mathrm{T}} \lfloor b \rfloor : c - A^{\mathrm{T}}\omega \geqslant \mathbf{0} \right\}. \tag{9.19}$$

Proof We show that all assumptions of Theorem 9.12 hold. First, let's show that Slater's condition (9.18) holds. On one hand, we have

$$\begin{aligned} \text{ri}\left(\text{conv}_{\mathbb{R}^m}\left(g^+\left(\mathbb{Z}_+^n\right) + \mathbb{Z}_+^m\right)\right) &= \text{ri}\left(\text{conv}_{\mathbb{R}^m}\left(g^+\left(\mathbb{Z}_+^n\right)\right)\right) + \text{ri}\left(\text{conv}_{\mathbb{R}^m}\left(\mathbb{Z}_+^n\right)\right) \\ &= \text{ri}\left(\text{conv}_{\mathbb{R}^m}\left(g^+\left(\mathbb{Z}_+^n\right)\right)\right) + (0,\infty)^n. \end{aligned}$$

On the other hand for $A \in \mathbb{Z}^{m\times n}$ and $x \in \mathbb{Z}^n_+$ we always have $g^+(x) = Ax - \lfloor b \rfloor$. This along with [15, Lemma 1.18] implies

$$\begin{aligned}\operatorname{ri}\left(\operatorname{conv}_{\mathbb{R}^m}\left(g^+\left(\mathbb{Z}^n_+\right)\right)\right) &= \operatorname{ri}\left(\operatorname{conv}_{\mathbb{R}^m}\left(A\left(\mathbb{Z}^n_+\right) - \lfloor b \rfloor\right)\right) = A\operatorname{ri}\left(\mathbb{R}^n_+\right) - \lfloor b \rfloor \\ &= A\,(0,\infty)^n - \lfloor b \rfloor\,.\end{aligned}$$

This means the Slater's condition (9.18) reduces to

$$0 \in \operatorname{ri}\left(\operatorname{conv}_{\mathbb{R}^m}\left(g^+\left(\mathbb{Z}^n_+\right) + \mathbb{Z}^m_+\right)\right) = A\,(0,\infty)^n - \lfloor b \rfloor + (0,\infty)^n\,,$$

which already holds since $Ax^1 < \lfloor b \rfloor$ for $x^1 \in (0,\infty)^n$. Consequently, all assumptions of Theorem 9.12 hold. Combining Theorems 9.11 and 9.12 we have (9.19). This completes the proof.

Hereafter, we discuss the relationship between discrete convexity and integrality. We say a polyhedron

$$P(A,b) = \left\{x : x \in \mathbb{R}^n_+ \quad \text{and} \quad Ax \leqslant b\right\}$$

is integral if

$$P(A,b) = \operatorname{conv}_{\mathbb{R}^n}\left(P(A,b) \cap \mathbb{Z}^n\right).$$

It is already known that the integrality of $P(A,b)$ and total unimodularity of A are equivalent. Let us recall the next definition.

Definition 9.14 Let A be an $m \times n$-matrix. The matrix A is called unimodular if all entries of A are integral and each nonsingular $m \times m$-submatrix of A has determinant ± 1. The matrix A is called totally unimodular if each square submatrix of A has determinant ± 1 or 0.

Theorem 9.15 ([18]) *Let A be an $m \times n$ integral matrix. Then, A is totally unimodular if and only if for every integral n-vector b, the polyhedron $\{x : x \geqslant 0$ and $Ax \leqslant b\}$ is integral.*

Next, we show that total unimodularity is a sufficient condition for convexity of function p_{ILP} defined by (9.17).

Lemma 9.16 *If $A \in \mathbb{Z}^{m\times n}$ is a totally unimodular matrix, then the function p_{ILP} defined by (9.17) is convex on $\mathbb{Z}^m$.*

Proof For a unimodular matrix A we have $g^+(x) = \lceil Ax - b \rceil = Ax - \lfloor b \rfloor$ and then

$$p_{\mathrm{ILP}}(y) = \inf\left\{c^{\mathrm{T}}x : x \in \mathbb{Z}^n_+ \quad \text{and} \quad Ax - \lfloor b \rfloor \leqslant y\right\}, \quad \text{for} \quad y \in \mathbb{Z}^m.$$

By the integrality of polyhedron $\left\{x \in \mathbb{R}^n_+ : Ax - \lfloor b \rfloor \leqslant y\right\}$ for $y \in \mathbb{Z}^m$ we have

$$\inf\left\{c^{\mathrm{T}}x : x \in \mathbb{Z}^n_+ \quad \text{and} \quad Ax - \lfloor b \rfloor \leqslant y\right\} = \inf\left\{c^{\mathrm{T}}x : x \in \mathbb{R}^n_+ \quad \text{and} \quad Ax - \lfloor b \rfloor \leqslant y\right\}.$$

The proof follows from the convexity of the function

$$g(y) = \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{R}_+^n \quad \text{and} \quad Ax - \lfloor b \rfloor \leqslant y \right\}$$

and from Theorem 4.18.

Combining Theorem 9.13 and above Lemma 9.16 we have the next result.

Theorem 9.17 (Total unimodularity and strong duality) *Let $A \in \mathbb{Z}^{m\times n}$ be a totally unimodular matrix and $b \in \mathbb{R}^m$. Consider the following primal problem*:

$$v(\mathrm{ILP}) := \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{Z}_+^n \quad \text{and} \quad Ax - b \leqslant \mathbf{0} \right\}.$$

If there exist $x^0 \in \mathbb{Z}_+^n$ and $x^1 \in (0,\infty)^n$ such that $Ax^0 = \lfloor b \rfloor$ and $Ax^1 < \lfloor b \rfloor$, then we have,

$$\infty > v\,(\mathrm{ILP}) = v\,(\mathrm{DILP}) = \sup_{\omega \in \mathbb{R}_-^m} \left\{ \omega^{\mathrm{T}} \lfloor b \rfloor : c - A^{\mathrm{T}} \omega \geqslant \mathbf{0} \right\}.$$

Example 9.18 Consider the following problem:

$$\begin{aligned} \min\ & -(x_1 + x_2) \\ \text{s.t.}\ & x_1 + 4x_2 \leqslant 19, \\ & 4x_1 + x_2 \leqslant 19, \\ & x_1, x_2 \in \mathbb{Z}_+. \end{aligned}$$

Obviously, this problem has the optimal solution $x^* = (3, 4)$ with the optimal value -7. By Theorem 9.11, we obtain its dual problem

$$\begin{aligned} \max\ & 19\omega_1 + 19\omega_2 \\ \text{s.t.}\ & \omega_1 + 4\omega_2 \leqslant -1, \\ & 4\omega_1 + \omega_2 \leqslant -1, \\ & \omega_1, \omega_2 \leqslant 0. \end{aligned}$$

The dual problem has the optimal solution $w^* = (-1/5, -1/5)$ with the optimal value $-38/5$. There is a positive duality gap between optimal values of primal and dual problems. However, the primal problem is equivalent to the following problem:

$$\begin{aligned} \min\ & -(x_1 + x_2) \\ \text{s.t.}\ & x_1 + x_2 \leqslant \tfrac{38}{5}, \\ & x_1 \leqslant \tfrac{19}{4}, \\ & x_2 \leqslant \tfrac{19}{4}, \\ & x_1, x_2 \in \mathbb{Z}_+. \end{aligned}$$

Notice that, in this case,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \frac{38}{5} \\ \frac{19}{4} \\ \frac{19}{4} \end{bmatrix}$$

satisfy the conditions in Theorem 9.17. By Theorem 9.11, we know that the problem has a dual:

$$\begin{aligned} \max\ & 7\omega_1 + 4\omega_2 + 4\omega_3 \\ \text{s.t.}\ & \omega_1 + \omega_2 \leqslant -1, \\ & \omega_1 + \omega_3 \leqslant -1, \\ & \omega_1, \omega_2, \omega_3 \leqslant 0, \end{aligned}$$

which has the optimal solution $\omega^* = (-1, 0, 0)$ with the optimum value -7. In this way, the duality gap vanishes.

The next example shows that total unimodularity is not a necessary condition for convexity of p_{ILP} or for strong duality.

Example 9.19 Consider the following problem:

$$\begin{aligned} \min\ & -x_1 \\ \text{s.t.}\ & x_1 + x_2 \leqslant 4, \\ & -x_1 + x_2 \leqslant 2, \\ & x_1, x_2 \in \mathbb{Z}_+. \end{aligned}$$

Obviously, this problem has the optimal solution $x^* = (4, 0)$ with the optimal value -4. By Theorem 9.11, we obtain its dual problem

$$\begin{aligned} \max\ & 4\omega_1 + 2\omega_2 \\ \text{s.t.}\ & \omega_1 - \omega_2 \leqslant -1, \\ & \omega_1 + \omega_2 \leqslant 0, \\ & \omega_1, \omega_2 \leqslant 0. \end{aligned}$$

The dual problem has the optimal solution $w^* = (-1, 0)$ with the optimal value -4. Notice that we have no duality gap despite the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

is not totally unimodular. Using the similar procedure in the proof of Lemma 9.16 and the equality

$$\begin{aligned} & \inf \left\{ c^{\mathrm{T}} u : u \in \mathbb{R}_+^2 \quad \text{and} \quad Au - b \leqslant y \right\} \\ = & \inf \left\{ c^{\mathrm{T}} x : x \in \mathbb{Z}_+^2 \quad \text{and} \quad Ax - b \leqslant y \right\} \quad \text{for} \quad y \in \mathbb{Z}^m, \end{aligned}$$

where $c^{\mathrm{T}} = (-1, 0)$, it is straightforward to show that $p_{\mathrm{ILP}}(y)$ is convex and all conditions in Theorem 9.13 hold.

In the following example, we reformulate the given ILP to get a convex function p_{ILP} on $\mathbb{Z}^2$.

Example 9.20 Consider the following problem:

$$v\,(\mathrm{ILP}) = \min_{t \in \mathbb{Z}_+} \{-2t : 2t \leqslant 5.3\}\,.$$

Obviously, this problem has a solution $t^* = 2$ with an optimal value -4. The primal problem is equivalent to the following ILP

$$v\,(\mathrm{ILP}) := \min_{t \in \mathbb{Z}_+} \{-2t : 2t \leqslant 5\}\,. \tag{9.20}$$

By Theorem 9.11, we get the dual problem of (9.20) as follows:

$$v\,(\mathrm{DILP}) = \max_{\omega \in \mathbb{R}_-} \{5\omega : 2\omega \leqslant -2\}\,.$$

The dual problem has the optimal solution -1 with the optimal value -5. There is a positive duality gap between $v\,(\mathrm{ILP})$ and $v\,(\mathrm{DILP})$. For $g^+\,(t) = \lceil 2t - 5.3 \rceil$, the function P_{ILP} defined by (9.17) becomes

$$p_{\mathrm{ILP}}\,(y) = \inf_{t \in \mathbb{Z}_+} \{-2t : 2t - 5 \leqslant y\} = -2 \left\lfloor \frac{y+5}{2} \right\rfloor \quad \text{for} \quad y = -5, -4, \cdots .$$

Obviously, P_{ILP} is not convex on $\mathbb{Z}$. If we add the new constraint $t \leqslant 2$ and the new slack variable $s \in \mathbb{Z}_+$ to the constraint $2t \leqslant 5.3$, then the new problem

$$\widetilde{v(\mathrm{ILP})} := \min_{t,s \in \mathbb{Z}_+} \{-2t : 2t + s \leqslant 5 \quad \text{and} \quad t \leqslant 2\} \tag{9.21}$$

is equivalent to the original problem. Note that the inequality $t \leqslant 2$ can also be derived by using the disjunctive procedure (see [9, pp. 212–213]) since the inequality $2t \leqslant 5.3$ implies the validity of the inequalities

$$t - \alpha\,(t - \delta) \leqslant 2$$

and

$$t + \beta\,(t - 1 - \delta) \leqslant 2$$

for $\alpha = 1$, $\delta = 2$, and $\beta = \frac{1.3}{0.7}$. Now, for

$$\widetilde{g}^+\,(t, s) = \begin{bmatrix} 2t + s - 5 \\ t - 2 \end{bmatrix},$$

we have

$$\mathbf{0} = \widetilde{g}^{+}(2,1) \in \widetilde{g}^{+}\left(\mathbb{Z}_{+}^{2}\right)$$

and

$$\widetilde{p_{\text{ILP}}}(y) := \inf_{(t,s)\in\mathbb{Z}_{+}^{2}} \left\{-2t : \begin{array}{c} 2t+s-5 \leqslant y_1 \\ t-2 \leqslant y_2 \end{array}\right\},$$

where $y = (y_1, y_2) \in \operatorname{dom}\widetilde{p_{\text{ILP}}} = \widetilde{g}^{+}\left(\mathbb{Z}_{+}^{2}\right) + \mathbb{Z}_{+}^{2}$. Obviously,

$$\mathbf{0} \in \operatorname{cri}_{\mathbb{Z}^2}\left(\widetilde{g}^{+}\left(\mathbb{Z}_{+}^{2}\right) + \mathbb{Z}_{+}^{2}\right) = ([-4,\infty) \times [-1,\infty)) \cap \mathbb{Z}^2,$$

and

$$\widetilde{p_{\text{ILP}}}(y) = -2(y_2 + 2).$$

This means, $\widetilde{p_{\text{ILP}}}$ is convex on $\mathbb{Z}^2$. Hence, all assumptions of Theorem 9.12 hold. The dual of the new problem (9.21) becomes

$$\begin{array}{ll} \max & 5\omega_1 + 2\omega_2 \\ \text{s.t.} & 2\omega_1 + \omega_2 \leqslant -2, \\ & \omega_1 \geqslant 0, \\ & \omega_1, \omega_2 \leqslant 0, \end{array}$$

which has an optimal solution $\{\omega_1 = 0, \omega_2 = -2\}$ with the optimal value -4. Hence, the duality gap vanishes.

Notice that if we add an unnecessary constraint to the problem, then the reformulated problem may not satisfy one of the conditions of Theorem 9.12. For example, let's add the slack variable $s \in \mathbb{Z}_+$ and the redundant constraint $t \leqslant 1$ to the primal problem.

Now, the primal problem becomes

$$\min_{t,s\in\mathbb{Z}_+} \{-2t : 2t + s \leqslant 5, \quad t \leqslant 1\}$$

for which Slater's condition does not hold.

Remark 9.21 Let $g^{+}(x) = \lceil Ax - b \rceil$. Theorem 9.12 shows that the convexity condition for the function p_{ILP} on $\mathbb{Z}^m$, the condition (9.12), and the Slater's condition of (9.18)

$$\mathbf{0} \in \operatorname{cri}_{\mathbb{Z}^m}\left(g^{+}\left(\mathbb{Z}_{+}^{n}\right) + \mathbb{Z}_{+}^{m}\right)$$

play essential roles to zero out the duality gap for integer linear programming problems. Theorem 9.13 and the results followed show that our strong duality theorem proposes a condition weaker than the integrality of $P(A, b)$. The conditions of discrete strong duality theorem (i.e., Theorem 9.12) may require a reformulation of ILP. The reformulation may include a new matrix, determined by some new constraints, which guarantees convexity of the function p_{ILP}. To be more precise, convexity condition for p_{ILP} and Slater's condition (9.18) may require adding new constraints to the primal problem while the condition (9.12) may require adding new slack variables to the given constraints. Hence, conditions of the discrete strong duality theorem could lead to an alternative procedure for the reformulation of ILP problems.

10 Conclusion

The objective of this paper is to establish a fundamental theory of convex analysis for the sets and functions over a discrete domain and develop a duality theory for solving optimization problems over discrete domains. We have successfully introduced a new notion of convexity for sets and functions over discrete domains, developed some discrete counterparts of the fundamental properties of conventional convex analysis and extended the classical duality concept for handling integer linear programming problems. This work has brought a new framework to study the fundamentals of convexity and duality over discrete domains.

Using the proposed new framework, we have shown a discrete analogue of the Fenchel–Moreau theorem to characterize the dual of the dual function. We have also derived a discrete version of Slater's condition that implies the strong duality theorem for integer linear programming and discussed the relationship between the new convexity notion and integrality. Unlike the known result of [10, Theorem 8.59], our strong duality theorem does not require M-convexity of the feasible domain and objective function. Actually, we have shown that strong duality holds even when M-convexity is violated in the new setting.

One particularly interesting result obtained in the new framework is that the dual of an integer linear program becomes a regular linear programming problem over a continuous domain of $\mathbb{R}^n$. This provides a possibility of using conventional optimization methods to solve (or estimate) an integer linear programming problem from the dual side. However, finding easier and more verifiable conditions to assert the discrete version of strong duality theorem remains a topic for further investigation. After all, solving integer linear programming problems is still NP-hard.

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