LECTURE 3: OPTIMALITY CONDITIONS

1. First order and second order information
2. Necessary and sufficient conditions of optimality
3. Convex functions

General setting

- General form nonlinear programming problem

$$\begin{aligned} \text{Min} \quad & f(x) \\ \text{s. t.} \quad & x \in S \subset E^n \end{aligned}$$

where S can be a "simple" set

$$\text{or} \quad S \triangleq \{x \in E^n \mid g_i(x) \le 0,\ i = 1, \ldots, m; \\ h_j(x) = 0,\ j = 1, \ldots, n; \\ x \in X\}$$

Local minimum

Definition A point $x^*\in S$ is said to be a *relative minimum point or a local minimum point* of f over S if there is an $\epsilon > 0$ such that $f(x) \geq f(x^*)$ for all $x \in S \cap N(x^*, \epsilon)$, where $N(x^*, \epsilon)$ is the neighborhood of x^* of radius ϵ. If $f(x) > f(x^*)$ for all $x \in S \cap N(x^*, \epsilon)$ and $x \neq x^*$, then x^* is said to be a *strictly relative minimum point* of f over S.

Global minimum

Definition A point $x^* \in S$ is said to be a *global minimum point* of f over S if $f(x) \geq f(x^*)$ for all $x \in S$. If $f(x) > f(x^*)$ for all $x \in S$, $x \neq x^*$, then x^* is said to be a *strictly global minimum point* of f over S.

Comments

- We always intend to seek a global minimum when formulating an optimization problem.
- In most situations, optimization theory and methodologies only enable us to locate local minimums.
- Global optimality can be achieved when certain convexity conditions are imposed.

A general iterative scheme

- A general scheme of an iterative solution procedure:

  Step 1: Start from a feasible solution *x* in S.

  Step 2: Check if the current solution is optimal.
  If the answer is Yes, stop.
  If the answer is No, continue.

  Step 3: Move to a better feasible solution and
  return to Step 2.

What are the feasible moves that lead to a better solution?

- Feasible direction
  - Along any given direction, the objective function can be regarded as a function of a single variable.
  - Given $x \in S \subset E^n$, a vector $d \in E^n$ is a *feasible direction* at x if there is an $\overline{\alpha} > 0$ such that $x + \alpha d \in S$ for all α, $0 \leq \alpha \leq \overline{\alpha}$.
  - A feasible direction is a good direction, if the objective function is reduced along the direction.

How do we know we have attained a minimum solution?

- First order necessary condition

- **Proposition.** Let S be a subset of E^n and let $f \in C^1$ be a function on S. If x^* is a relative minimum point of f over S, then for any $d \in E^n$ that is a feasible direction at x^*, we have $\nabla f(x^*)d \geq 0$.

- **Corollary (Unconstrained case).** Let S be a subset of E^n and let $f \in C^1$ be a function on S. If x^* is a relative minimum point of f over S and if x^* is an interior point of S, then $\nabla f(x^*) = 0$.

Example 1

Example: Constrained problem:

$$\begin{aligned} \min \quad & f(x_1, x_2) = x_1^2 - x_1 + x_2 + x_1 x_2 \\ \text{s. t.} \quad & x_1, x_2 \geq 0 \end{aligned}$$

Check if $x^* = [1/2, 0]$ satisfies the first-order necessary condition or not.

$$\begin{aligned} \nabla f(x) \,|_{x^*} &= [2x_1 - 1 + x_2, 1 + x_1]\big|_{x_1=1/2, x_2=0} \\ &= [0, 3/2] \end{aligned}$$

$\Rightarrow \nabla f(x^*)d \geq 0$ for all d with $d_2 \geq 0$ (feasible direction at x^*).

Example 2

Example: Unconstrained problem:

$$\min f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 - 3x_2$$

Global minimum is known at $x_1 = 1$, $x_2 = 2$.

At this point,

$$\nabla f(x) = [2x_1 - x_2,\ -x_1 + 2x_2 - 3]$$

$$= [0, 0]$$

Comments

- The necessary conditions in the pure unconstrained case lead to a system of *n* equations in *n* unknowns.

- Is the condition a sufficient condition? Why?

- How about the condition of

$$\nabla f(x^*)d > 0?$$

Proof of the proposition

If $\exists$ a feasible direction $d \in E^n$ at x^* with $\nabla f(x^*)d < 0$, then $\exists\ \bar{\alpha} > 0$ s.t. $x(\alpha) = x^* + \alpha d \in S$ with $0 < \alpha < \bar{\alpha}$ and

$$
\begin{aligned}
f(x(\alpha)) &= f(x^*) + \nabla f(x^*)(x(\alpha) - x^*) + O(\alpha^2) \\
&= f(x^*) + \alpha \nabla f(x^*)d + O(\alpha^2) \\
&< f(x^*)\ , \quad \text{if } \alpha \text{ is sufficiently small.}
\end{aligned}
$$

This contradicts to the fact that x^* is a local minimum point of f over S.

Corollary – Variational Inequalities

- **Proposition:** Let $S \subset E^n$ be convex and $f : E^n \to R$ be $C^1(S)$. If x^* is a relative minimum point of f over S, then x^* is a solution of the following variational inequality problem:

$$
\begin{aligned}
& \text{Find} \quad x \in S \\
(VI) \quad & \text{s. t.} \quad \langle x' - x, \nabla f(x) \rangle \geq 0, \\
& \qquad\qquad \forall\, x' \in S.
\end{aligned}
$$

Second order conditions

Proposition (Second-order necessary conditions). Let S be a subset of E^n and let $f \in C^2$ be a function on S. If x^* is a relative minimum point of f over S, then for any $d \in E^n$ that is a feasible direction at x^*, we have

(i) $\nabla f(x^*)d \geq 0$.

(ii) if $\nabla f(x^*)d = 0$, then $d^T \nabla^2 f(x^*)d \geq 0$.

Proof:

$$\begin{aligned} f(x(\alpha)) &= f(x^*) + \tfrac{1}{2}(x(\alpha) - x^*)^T \nabla^2 f(x^*)(x(\alpha) - x^*) + O(\alpha^3) \\ &= f(x^*) + \tfrac{1}{2}\alpha^2 d^T \nabla^2 f(x^*)d + O(\alpha^3) \end{aligned}$$

Example 3

Example: Constrained problem:

$$\begin{aligned} \min \quad & f(x_1, x_2) = x_1^2 - x_1 + x_2 + x_1 x_2 \\ \text{s. t.} \quad & x_1, x_2 \geq 0 \end{aligned}$$

Check if $x^* = [1/2, 0]$ satisfies the second-order necessary condition or not.

$\nabla f(x) \mid_{x^*} = [0, 3/2]$, since $\nabla f(x^*) d = 3/2 d_2 = 0$

$\Rightarrow d_2 = 0$

$\Rightarrow d^T \nabla^2 f(x^*) d = 2 d_1^2 \geq 0$

Second order necessary condition

- **Proposition (Second-order necessary conditions – unconstrained case).** Let x^* be an interior point of the set S, and suppose x^* is a relative minimum point of $f \in C^2$. Then

  (i) $\nabla f(x^*) = 0$.

  (ii) $F(x^*)$ is positive semidefinite.

Example 4

Example: Unconstrained problem:

$$\min f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 - 3x_2$$

Global minimum is known at $x_1 = 1$, $x_2 = 2$.

At this point,

$$\nabla f(x) = [2x_1 - x_2, \ -x_1 + 2x_2 - 3]$$

$$= [0, 0]$$

and $F(x)$ is positive definite.

Example 5

Example: Constrained problem:

$$\begin{array}{ll} \min & f(x_1, x_2) = x_1^3 - x_1^2 x_2 + 2x_2^2 \\ \text{s. t.} & x_1, x_2 \geq 0 \end{array}$$

$x^* = [6, 9]$ is a solution to the first-order necessary condition:

$$\nabla f(x) \mid_x = [3x_1^2 - 2x_1 x_2, -x_1^2 + 4x_2] = 0$$

But, x^* does not satisfy the second-order necessary condition,

$$F = \begin{bmatrix} 6x_1 - 2x_2 & -2x_1 \\ -2x_1 & 4 \end{bmatrix} \Big|_{x^*} = \begin{bmatrix} 18 & -12 \\ -12 & 4 \end{bmatrix}$$

Second order sufficient condition

- **Proposition (Second-order sufficient conditions – unconstrained case).** Let $f \in C^2$ be a function on a region in which the point x^* is an interior point. Suppose in addition that

  (i) $\nabla f(x^*) = 0$.

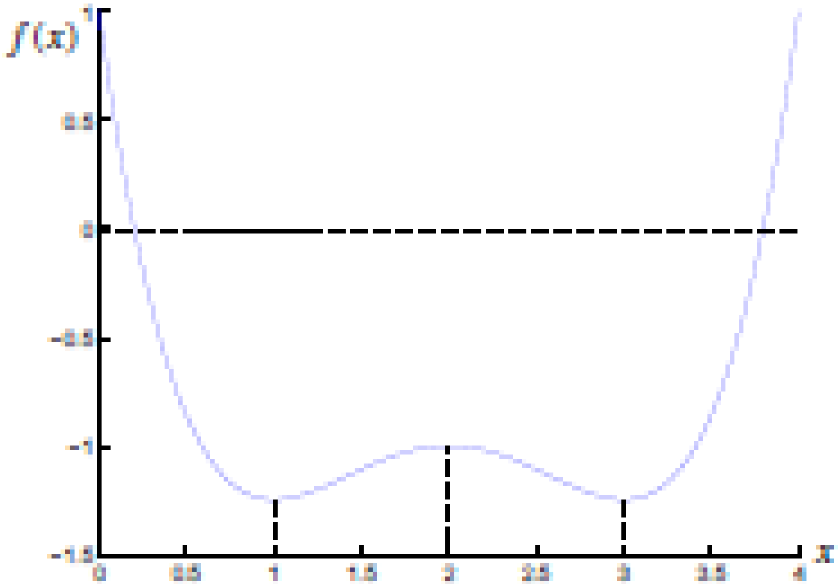
  (ii) $F(x^*)$ is positive definite.

  Then x^* is a strictly relative minimum point of f.

Example 6

$$\text{Min} \quad f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x + 1$$

$$\text{s. t.} \qquad 0 \le x \le 4.$$



Continue

- First-order information:

$$f'(x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3).$$

$$f'(0) = -6,\ f'(1) = f'(2) = f'(3) = 0,\ f'(4) = 6.$$

- Second-order information:

$$f''(x) = 3x^2 - 12x + 11$$

$$\Rightarrow\ f''(1) > 0,\ f''(2) < 0,\ f''(3) > 0.$$

By checking the 1st-order necessary conditions, only $x = 1$, $x = 2$ and $x = 3$ are satisfied.

By checking the 2nd-order necessary conditions, only $x = 1$ and $x = 3$ are satisfied.

By checking the 2nd-order sufficient conditions, we know $x^* = 1$ or 3 with $f(x^*) = -1.25$.

Convex functions - definition

- Let $\Omega \subset E^n$ be a convex set and $f : \Omega \rightarrow R$ be a real-valued function. Then f is convex on Ω, if

$$f(\alpha x^1 + (1-\alpha)x^2) \;\leq\; \alpha f(x^1) + (1-\alpha) f(x^2)$$

$\forall\, x^1, x^2 \in \Omega$ and $\alpha \in [0, 1]$.

Moreover, f is strictly convex on Ω, if

$$f(\alpha x^1 + (1-\alpha)x^2) \;<\; \alpha f(x^1) + (1-\alpha) f(x^2)$$

$\forall\, x^1 \neq x^2,\;\; x^1, x^2 \in \Omega$ and $\alpha \in (0, 1)$.

Concave functions

- $g : \Omega \longrightarrow R$ is (strictly) concave on Ω, if $f = -g$ is (strictly) convex on Ω.

Graph and epigraph of a function

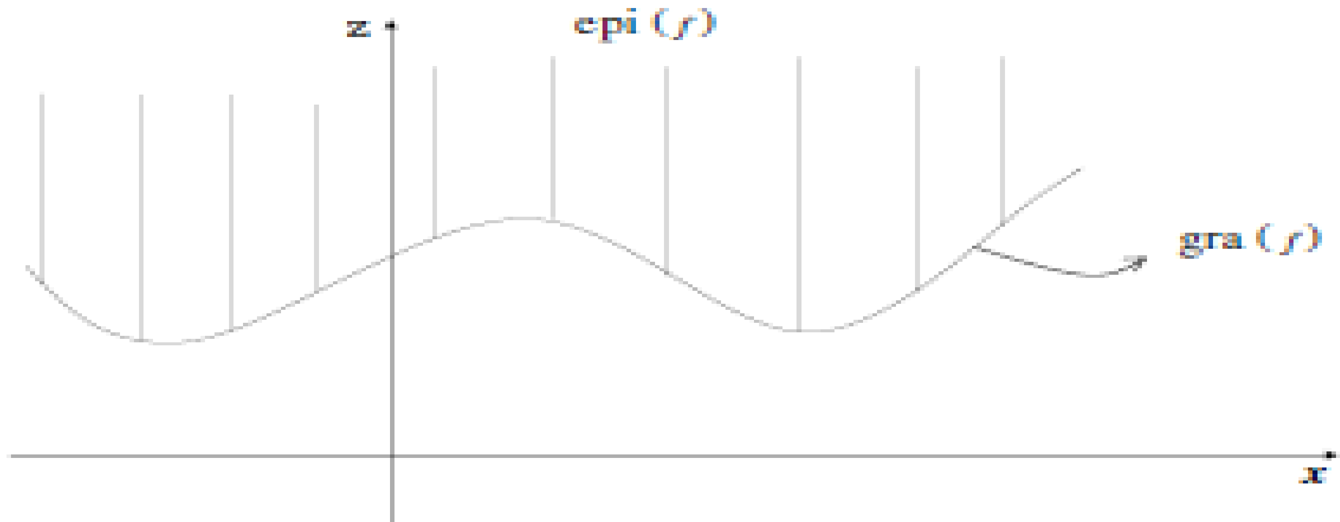
- Let $\Omega \subset E^n$ and $f : \Omega \to R$.

  The graph of f is

$$gra(f) \triangleq \{(x, z) \in E^{n+1} \mid x \in \Omega \ and \ f(x) = z\}$$

  The epigraph of f is

$$epi(f) \triangleq \{(x, z) \in E^{n+1} \mid x \in \Omega \ and \ f(x) \leq z\}$$



Set based definition of convex functions

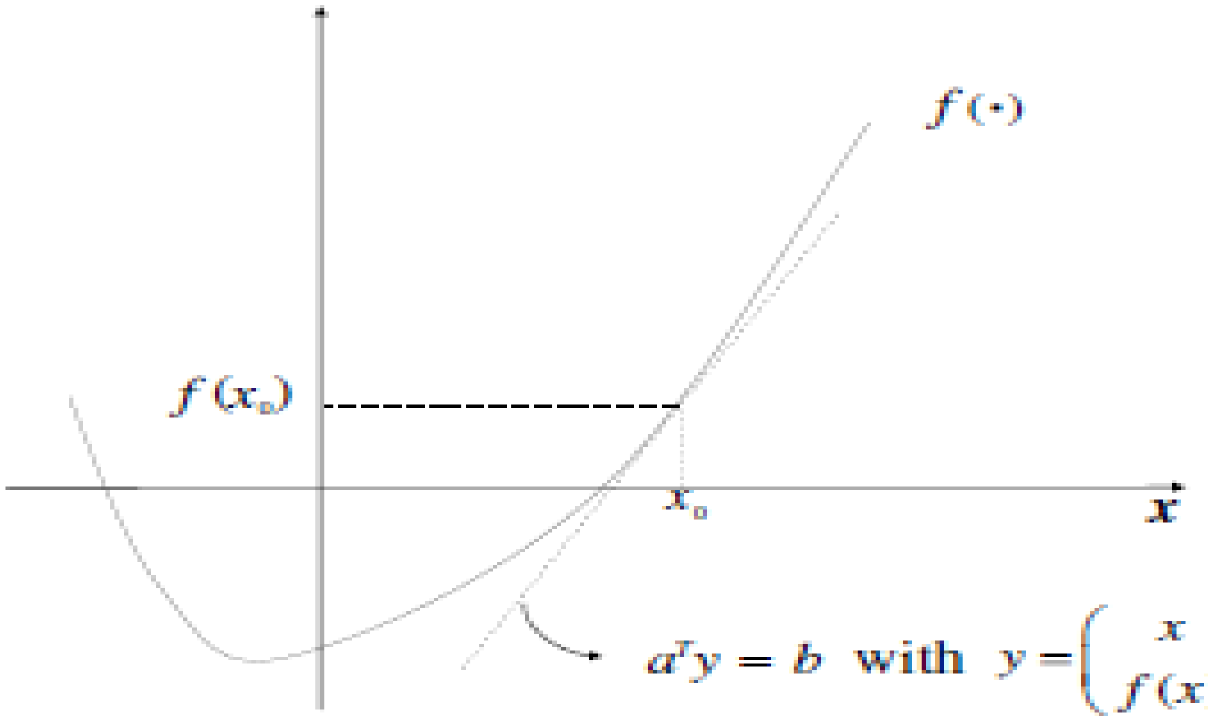
- Definition

- A function $f : \Omega \subset E^n \to R$ is convex if $epi(f)$ is a convex subset of E^{n+1}.

- **Theorem:**
  For a convex function f, if each point in $gra(f)$ is an extreme point of $epi(f)$, then the function f is strictly convex.

Question

Let $f : \Omega \subset E^n \longrightarrow R$ be convex and $f \in C^1(\Omega)$.
For $x^0 \in \Omega$, what's the supporting hyperplane of $epi(f)$ at $(x^0, f(x^0))$

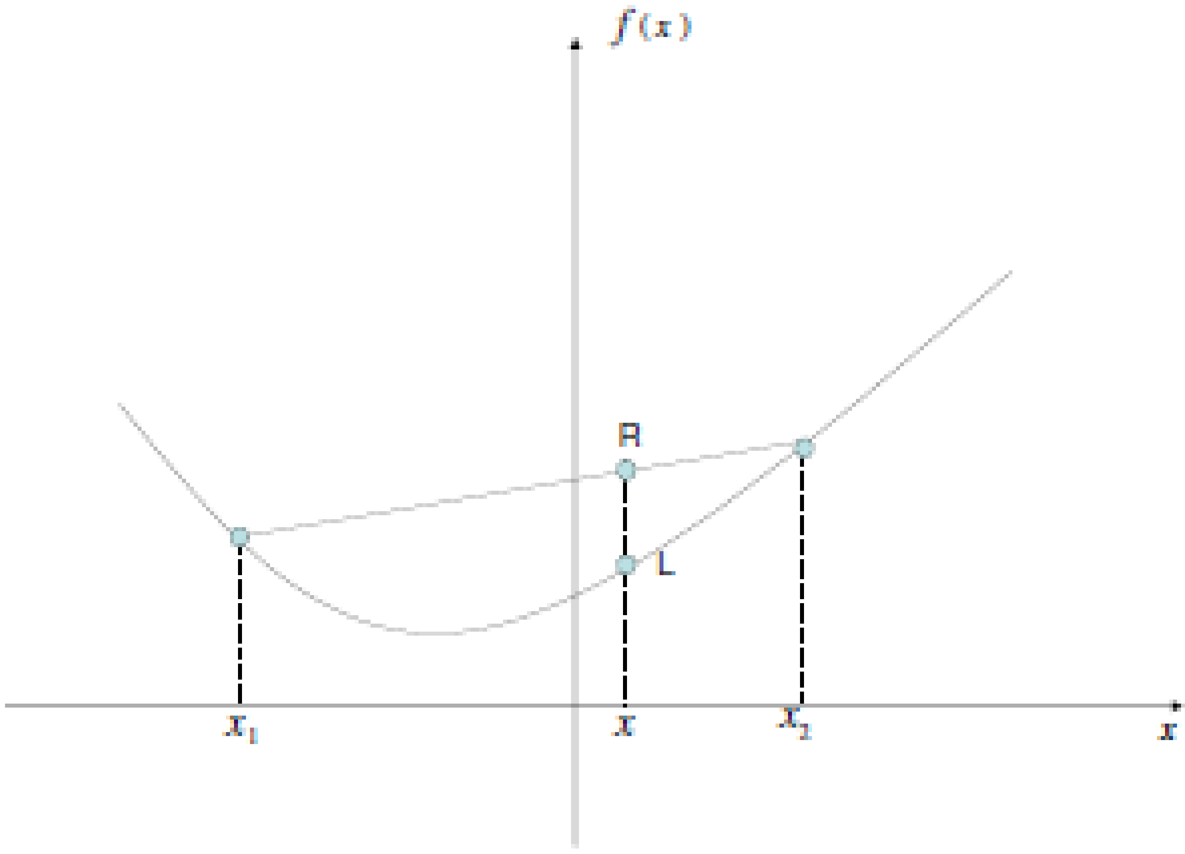


$$a^T y = b \quad \text{with} \quad y = \begin{pmatrix} x \\ f(x) \end{pmatrix}$$

$a = ?\quad b = ?$

Basic property - 1

- Overestimate by two-point information



Basic property - 2

- **Theorem:**

  Let f be a convex function on a convex set $\Omega \subset E^n$.

  Then

  $$f\left(\sum_{i=1}^{m} \alpha_i x^i\right) \leq \sum_{i=1}^{m} \alpha_i \, f(x^i)$$

  $\forall x^i \in \Omega, \quad \alpha_i \in [0,1] \quad and \quad \sum_{i=1}^{m} \alpha_i = 1$

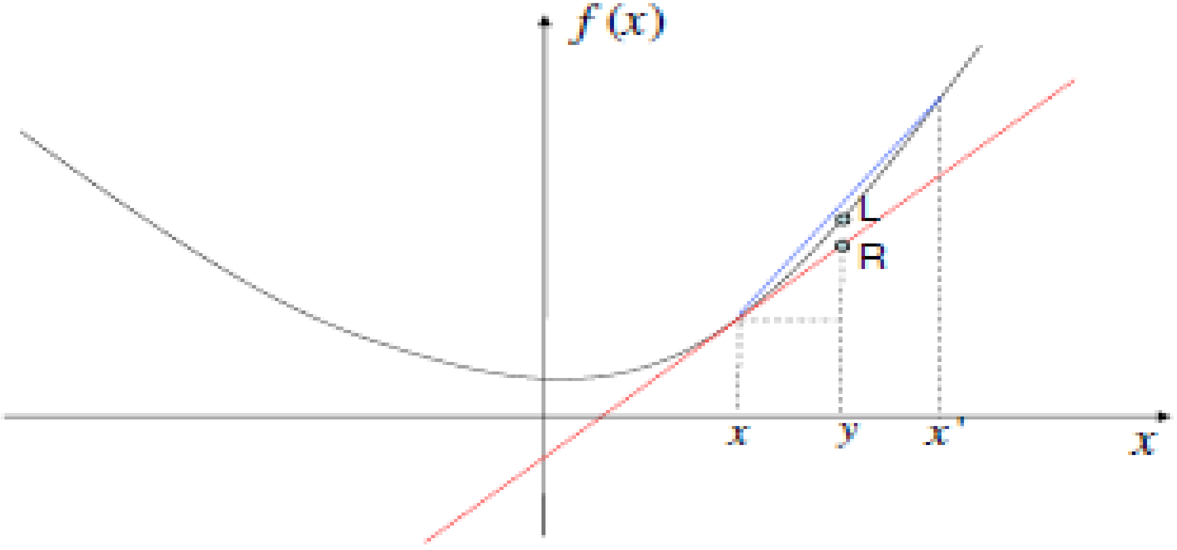
  (Jensen's inequality)

Basic property - 3

- <u>**Theorem:**</u>

  Let $f \in C^1$. Then f is convex on a convex set $\Omega \subset E^n$ if, and only if,

$$f(y) \geq f(x) + \nabla f(x)(y - x), \quad \forall\, x, y \in \Omega$$



(underestimate by one-point information)

Proof

($\Rightarrow$) If f is convex, then for $x, y \in \Omega$,

$f(\alpha y+(1-\alpha)x) \leq \alpha f(y)+(1-\alpha)f(x), \; \forall\alpha \in [0,1]$

For $\alpha \neq 0$,

$$\frac{f(x+\alpha(y-x))-f(x)}{\alpha} \leq f(y)-f(x)$$

As $\alpha \longrightarrow 0$, we have

$$\nabla f(x)(y-x) \leq f(y)-f(x)$$

Proof

$(\Leftarrow)$ Assume that

$$f(y) \;\geq\; f(x) + \nabla f(x)(y - x), \quad \forall x, y \in \Omega$$

Given $x^1, x^2 \in \Omega$, and any $\bar{\alpha} \in [0, 1]$.

Consider $\quad \bar{x} = \bar{\alpha}x^1 + (1 - \bar{\alpha})x^2$, then

$$f(x^1) \;\geq\; f(\bar{x}) + \nabla f(\bar{x})(x^1 - \bar{x})$$

$$f(x^2) \;\geq\; f(\bar{x}) + \nabla f(\bar{x})(x^2 - \bar{x})$$

Multiplying the first by $\bar{\alpha}$ and the second by $1 - \bar{\alpha}$ and adding up, we have

$$\begin{aligned}\bar{\alpha}f(x^1)+(1-\bar{\alpha})f(x^2) &\geq f(\bar{x})+\nabla f(\bar{x})(\bar{\alpha}x^1+(1-\bar{\alpha})x^2-\bar{x}) \\ &= f(\bar{\alpha}x^1 + (1-\bar{\alpha})x^2) + \nabla f(\bar{x})(0) \\ &= f(\bar{\alpha}x^1 + (1-\bar{\alpha})x^2)\end{aligned}$$

Basic properties - 4 and 5

- **Theorem:**

  Let $\Omega \subset E^n$ be a convex set, $f_1, f_2 : \Omega \to R$ be convex functions.

  Then (i) $f_1 + f_2$ is convex on Ω

  (ii) βf_1 is convex on Ω, $\forall\ \beta \geq 0$

- **Theorem:**

  Let f be a convex function on a convex set $\Omega \subset E^n$. Then the set

  $I_c \triangleq \{x \in \Omega \mid f(x) \leq c\}$ is convex, $\forall\ c \in R$.

Basic property - 6

- **Theorem:**

  Let $f \in C^2$ and $\Omega \subset E^n$ is convex with $int(\Omega) \neq \phi$. Then f is convex on Ω, if and only if, the Hessian matrix F is positive semidefinite over Ω.

Proof

By Taylor's Theorem,

$$f(y) = f(x) + \nabla f(x)(y - x) + \frac{1}{2}(y-x)^T F(x+\alpha(y-x))(y-x)$$

for some $\alpha \in [0, 1]$.

Additional properties

- **Theorem:**

  Let $S \subset E^n$ be convex and $f : S \to R$.

  Then f is (strictly) convex if, and only if, $g(s) \triangleq f(x^0 + sd)$ is (strictly) convex on $I \triangleq \{s \in R \,\big|\, x^0 + sd \in S\}$ for any given $x^0 \in S$ and $d \in E^n$.

- **Theorem:**

  Let f be (strictly) convex on $S \subset E^n$ and $x = My + b$ is an affine transformation from E^m to E^n. Then $g(y) \triangleq f(My + b)$ is (strictly) convex on $\{y \in E^m \,\big|\, My + b \in S\}$, if M has full rank.

Additional properties

- **Theorem:**

  Let f_j, $j = 1, \ldots, p$, be convex on $S \subset E^n$ and $\alpha_j \geq 0$. Then $f \triangleq \sum_{j=1}^{p} \alpha_j f_j$ is convex on S. In addition, if $\exists\ i$ such that f_i is strictly convex on S and $\alpha_i > 0$, then $f \triangleq \sum_{j=1}^{p} \alpha_j f_j$ is strictly convex on S.

Additional properties

- **Theorem:**
  Let f_j, $j = 1, 2, \ldots$, be convex on $S \subset E^n$.
  If $\lim_{j \to \infty} f_j(x)$ exists for each $x \in S$, then
  $f(x) \triangleq \lim_{j \to \infty} f_j(x)$ is convex on S.

- **Theorem:**
  Let Ω be an index set and $\{f_w \mid w \in \Omega\}$ be a family of convex functions on $S \subset E^n$.
  Then, $f(x) \triangleq \sup_{w \in \Omega} f_w(x)$ is convex on $\{x \in S \mid \sup_{w \in \Omega} f_w(x) < +\infty\}$. In addition, if Ω is finite and f_w is strictly convex for each $w \in \Omega$, then f is strictly convex on S.

Additional properties

- **Theorem:**

  Let f_1 be convex on $S_1 \subset E^n$ and f_2 be convex and non-decreasing on a set $T \supset f_1(S_1)$. Then the composition function $f_2 \circ f_1\ (x) \triangleq f_2(\ f_1(x)\ )$ is convex on S_1.

  In addition, if f_1 is strictly convex on S_1 and f_2 is increasing, then $f_2 \circ f_1$ is strictly convex on S_1.

Minimization of convex functions

- **Theorem:**

  Let f be a convex function defined on the convex set S. Then any relative minimum of f is a global minimum and the set τ where f achieves its minimum is convex.

Proof

(*i*) If $x^* \in \Omega$ is a local minimum and $\exists\ y \in \Omega$ with $f(y) < f(x^*)$, then

$$f(\alpha y + (1-\alpha)x^*) \le \alpha f(y) + (1-\alpha)f(x^*) < f(x^*)$$

for $\alpha \in (0,1)$

This contradicts to the fact that x^* is a local minimum.

(*ii*) $\tau = \{x \mid f(x) \le f(x^*), \quad x \in \Omega\}$ is obviously convex.

Sufficient and necessary conditions

- For convex functions, the first order necessary condition is also a sufficient condition.

- **Theorem:**

  Let $f \in C^1$ be convex on a convex set $\Omega \subset E^n$. If $\exists\, x^* \in \Omega$, s.t.

  $$\nabla f(x^*)(y - x^*) \geq 0, \quad \forall y \in \Omega$$

  then x^* is a global minimum of f over Ω

Proof

Proof: Since

$$f(y) \geq f(x^*) + \nabla f(x^*)(y - x^*) \geq f(x^*), \ \ \forall\, y \in \Omega,$$

and any $y \in \Omega$ can be reached from x^* along

a feasible direction $y - x^*$.

Example

- **Example:** Check the convexity of the following optimization problem and find its (global) minimum.

$$\min f(x_1, x_2, x_3) = 4x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + x_1x_3 - 3x_1 - 2x_2$$

Maximization of convex functions

- <u>Theorem:</u>

  Let f be a convex function defined on the bounded, closed convex set $\Omega \subset E^n$. If f achieves global maximum on Ω, then one maximizer falls in $bdry\,(\Omega)$.

Proof

Assume $x^* \in \Omega$ is a global maximizer of f. If x^* is not a boundary point of Ω, then

$$\exists\, x^1, x^2 \in \; bdry(\Omega)$$

s.t.

$$x^* = \alpha x^1 + (1-\alpha)x^2 \text{ for some } \alpha \in (0,1)$$

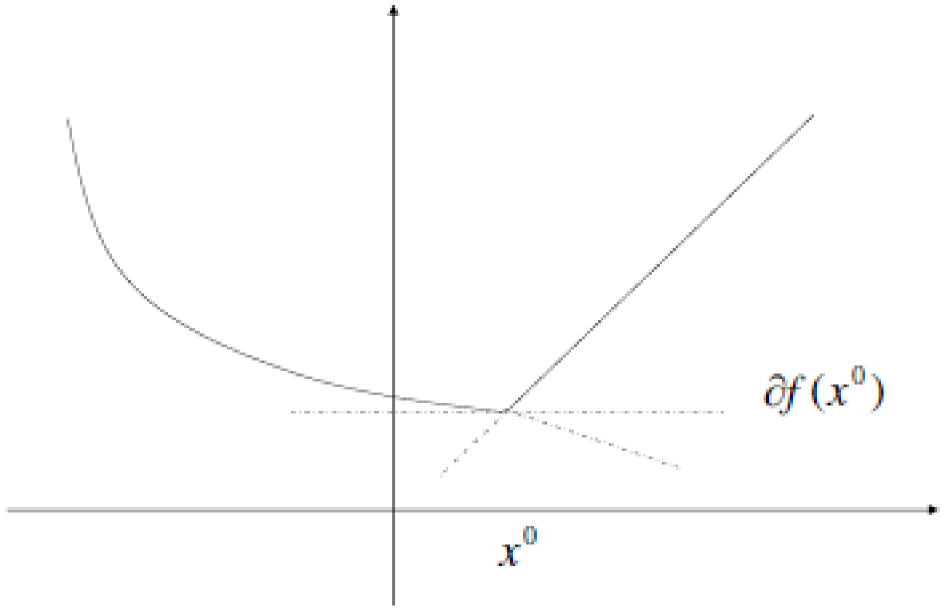
By convexity of f,

$$\begin{aligned} f(x^*) &\leq \alpha f(x^1) + (1-\alpha) f(x^2) \\ &\leq \max\{f(x^1), f(x^2)\} \end{aligned}$$

Therefore either x^1 or x^2 is a global maximizer.

Non-differentiable convex functions

- Where is the first order information?
  - subgradient and subdifferential



Subgradient and subdifferential

- Definition

A vector *y* is said to be a subgradient of a convex function *f* (over a set *S*) at a point x^0 if

$$f(x) \geq f(x^0) + \langle y, x - x^0 \rangle, \forall x \in S$$

- Definition

The set of all subgradients of *f* at x^0 iis called the subdifferential of *f* at x^0 and is denoted by

$$\partial f(x^0) = \{y \in E^n \mid f(x) \geq f(x^0) + \langle y, x - x^0 \rangle, \forall x \in S\}$$

Properties

1. The graph of the affine function

$$h(x) = f(x^0)+\langle y, x-x^0\rangle$$

is a non-vertical supporting hyperplane to the convex set *epi*(*f*) at the point of (x^0 , $f(x^0)$).

2. The subdifferential set $\partial f(x^0)$ is closed and convex.

3. $\partial f(x^0)$ can be empty, singleton, or a set with infinitely many elements. When it is not empty, *f* is said to be subdifferentiable at x^0.

4. $\nabla f(x^0) \in \partial f(x^0)$ if f is differentiable at x^0.

$\{\nabla f(x^0)\} = \partial f(x^0)$ if f is convex and differentiable at $x^0 \in int(S)$.

Examples

- In R, $f(x) = |x|$ is subdifferentiable at every point and

  $$\partial f(0) = [-1, 1].$$

- In E^n, the Euclidean norm $f(x) = ||x||$ is subdifferentiable at every point and $\partial f(0)$ consists of all the vectors y such that

  $$||x|| \geq \langle y, x \rangle \quad \text{for all } x.$$

  This means the Euclidean unit ball !